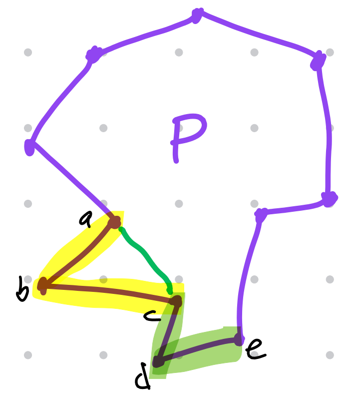
January 6, 2025

Important for HW1:

If a polygon has consecutive vertices a, b, c and $\overline{ac}$ is a diagonal, then a, b, c is called an ear.

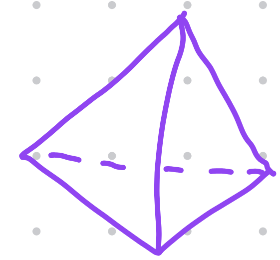


Cor. 19: Every polygon with more than 3 vertices has at least 2 ears with non-adjacent tips.

POLYHEDRA: TETRAHEDRALIZATION

TETRAHEDRON: a pyramid with a triangular base

has: 4 vertices, 6 edges, 4 triangular faces



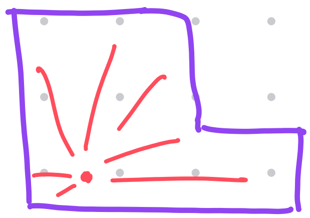
A TETRAHEDRALIZATION of a polyhedron is a partition of its interior into tetrahedra whose edges are diagonals of the polyhedron.

NOTES:

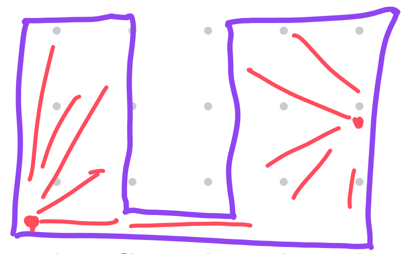
- Two different tetrahedralizations of the same polyhedron may have different numbers of tetrahedra.
- Not every polyhedron has a tetrahedralization!
  example: Schönhardt polyhedron

ART GALLERY PROBLEM

Suppose an art gallery has a floor plan modeled by a polygon. A guard occupies a single point and can see in straight lines to the gallery walls.



1 guard



2 guards

If the polygon has n sides, what is the smallest number of guards guaranteed to cover it entirely?

Art Gallery Problems

MATH 261 Computational Geometry

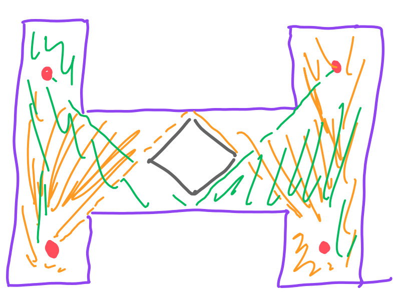
Suppose an art gallery has a floor plan modeled by a polygon. A guard occupies a single point and can see in straight lines in all directions to the gallery walls.

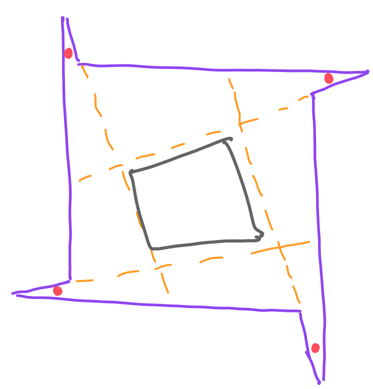
Question 1: If the polygon has n sides, what is the smallest number of guards guaranteed to protect the gallery? Try to find "bad" polygons – n-sided polygons that require "lots" of guards as a function of n.

$\left\lfloor \frac{n}{3} \right\rfloor$ guards are necessary for some n-gons and sufficient for all n-gons

"floor" function: round down to the nearest integer

Question 2: Suppose that a certain placement of guards can see all of the walls of a particular gallery. Can they necessarily see all of the interior?





Modification 1: Do your answers change if guards may only be stationed at vertices of the polygon?

Modification 2: Now assume that the art gallery is an *orthogonal polygon* — that is, all corners of the gallery are right angles. How does this change your answers for Questions 1 and 2?

Numbers of triangulations of convex n-gons:

$1, 2, 5, 14, 42, \ldots$

the Catalan numbers

edge contraction example

$t_5 = |\mathcal{T}_5| = 5$

number of edges that meet the vertex

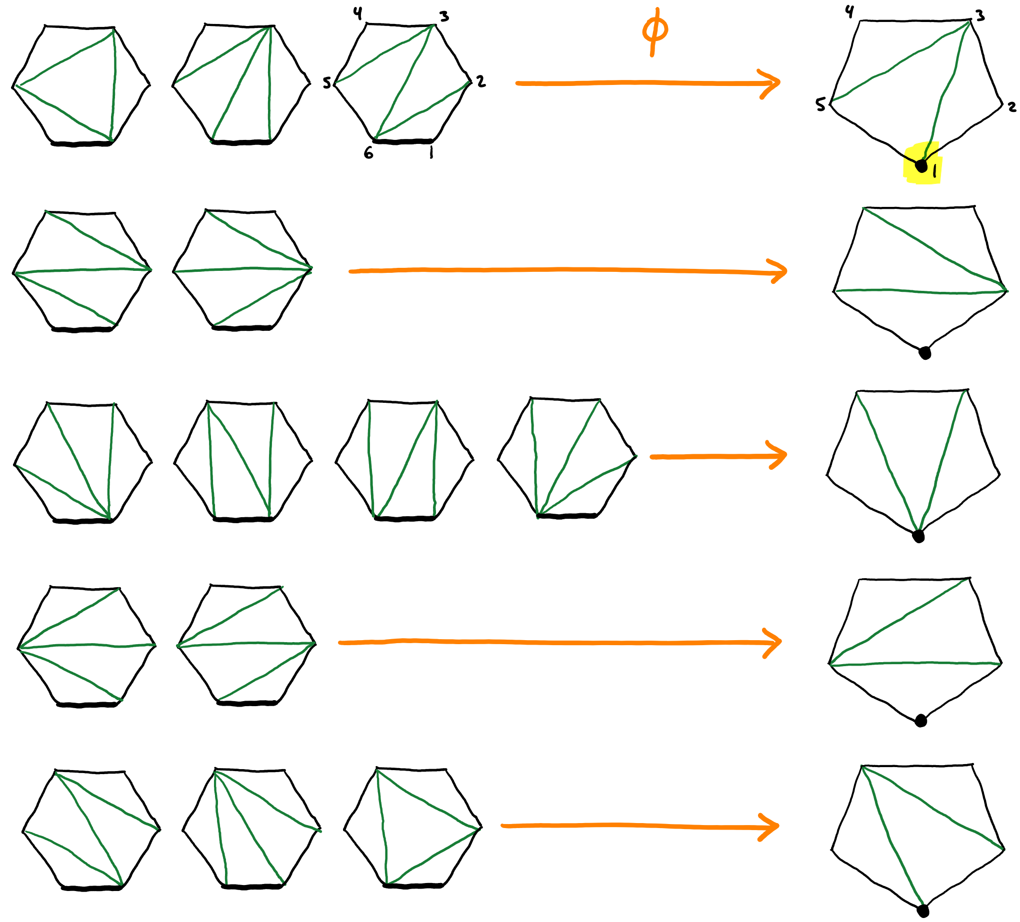
$t_6 = |\mathcal{T}_6|$

$t_6 = 14$

$\mathcal{T}_6$: set of triangulations of a convex hexagon

$\mathcal{T}_5$: set of triangulations of a convex pentagon

degree of vertex 1 in the pentagon:



3

2

4

2

3

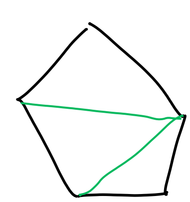
principle: The degree of vertex 1 in a triangulation of the pentagon equals the number of triangulations of the hexagon that ϕ maps to the triangulation of the pentagon.

Then: $$t_6 = \sum_{T \in \mathcal{T}_5} (\text{degree of vertex } 1 \text{ in } T)$$

Now sum over all vertices of T:

$$5 \cdot t_6 = \sum_{i=1}^{5} \sum_{T \in \mathcal{T}_5} (\text{degree of vertex } i \text{ in } T)$$

$$= \sum_{T \in \mathcal{T}_5} \underbrace{\boxed{\sum_{i=1}^{5} (\text{degree of vertex } i \text{ in } T)}}_{\text{sum of degrees of all vertices in a triangulation of the pentagon}} = \sum_{T \in \mathcal{T}_5} (14)$$



$5 t_6 = 14 t_5$

Now generalize:

consider $\phi : \mathcal{T}_{n+2} \longrightarrow \mathcal{T}_{n+1}$

number of triangulations of a convex $(n+2)$-gon

$$t_{n+2} = \sum_{T \in \mathcal{T}_{n+1}} (\text{degree of vertex } 1 \text{ in } T)$$

Sum over all vertices of T:

$$(n+1)\, t_{n+2} = \underbrace{\sum_{T \in \mathcal{T}_{n+1}}}_{t_{n+1}} \boxed{\sum_{i=1}^{n+1} (\text{degree of vertex } i \text{ in } T)}$$

total degree is twice the number of edges and diagonals in T

For $n > 1$:

$$(n+1)\, t_{n+2} = 2(2n-1)\, t_{n+1}$$

$$2(\underbrace{(n+1)}_{\text{edges}} + \underbrace{(n-2)}_{\text{diagonals}}) = 2(2n-1)$$

Also, $t_3 = 1$.

So: $$t_{n+2} = \frac{2(2n-1)}{n+1}\, t_{n+1}$$

also: $t_{n+1} = \dfrac{2(2n-3)}{n}\, t_n$

$$t_{n+2} = \frac{2(2n-1)}{(n+1)} \cdot \frac{2(2n-3)}{(n)} \cdot \frac{2(2n-5)}{(n-1)} \cdot \ldots \cdot \underbrace{\frac{2(3)}{3}}_{n=2} \underbrace{t_3}_{=1} \cdot \frac{2}{2}$$

$$t_{n+2} = \frac{1}{(n+1)!} \cdot \frac{2n(2n-1)(2(n-1))(2n-3)(2(n-2))(2n-5)\cdots(2\cdot 2)(3)(2)}{n \cdot (n-1) \cdot (n-2) \cdots \cdot 2}$$

$$\boxed{t_{n+2} = \frac{1}{(n+1)!} \cdot \frac{(2n)!}{n!}}$$

$$\boxed{t_{n+2} = \frac{1}{(n+1)} \binom{2n}{n}} = C_n$$

$\uparrow$ nth Catalan number

Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

"n choose k"

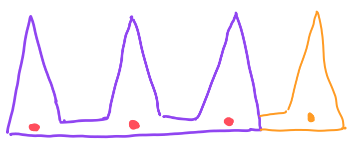
An Art Gallery Theorem

MATH 261 Computational Geometry

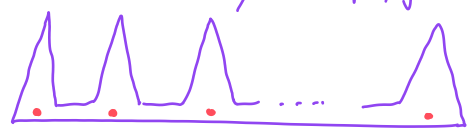
Theorem: To cover a polygon with n vertices, $\lfloor n/3 \rfloor$ guards are necessary for some polygons and sufficient for all of them.

Proof that $\lfloor n/3 \rfloor$ guards are necessary:
Describe a family of polygons that shows this.

example: n=9

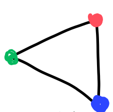


Consider the family of polygons:

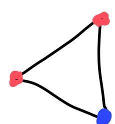


This polygon has 3n sides, n "spikes", and requires n guards.

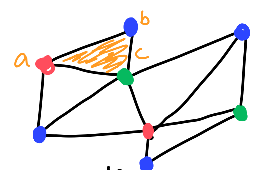
Proof that $\lfloor n/3 \rfloor$ guards are sufficient:
Let P be a polygon with n vertices. We know that P can be triangulated.
Show that each vertex of P can be assigned one of three colors so that any pair of vertices connected by an edge of P or a diagonal of the triangulation have different colors.



OK



not OK



P

OK

Base Case: If $n=3$, then P is a triangle, so color each vertex a different color.

Induction: Assume the statement holds for polygons of $n-1$ vertices.
Since $n>3$, we know that P has an ear, a,b,c.
Removing $\triangle abc$ produces a polygon P' with $n-1$ vertices.
By the inductive hypothesis, P' can be 3-colored.
Replace vertex b and color it differently than a and c.

To complete the proof, explain how the least-used vertex color gives a placement of at most $\lfloor n/3 \rfloor$ guards that cover the polygon.

Since each triangle has a vertex of each color, guards placed on all vertices of any specific color will cover the polygon.

The least used color appears on at most $\lfloor \frac{n}{3} \rfloor$ vertices optional problems to think about:

Choose one of the following problems to think about with your group.

Orthogonal polygons: How could you show that $\lfloor n/4 \rfloor$ guards are necessary for some and sufficient for all orthogonal polygons?

Algorithms: How could you program a computer to find a placement of at most $\lfloor n/3 \rfloor$ guards that cover a given polygon? Suppose that the polygon is specified by a list of (x, y) coordinates of its vertices, in order around the polygon.