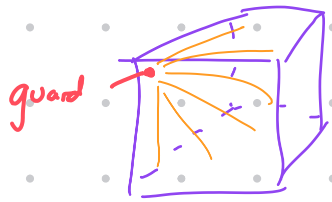
7 January 2025

Art gallery in 3D



POLYHEDRAL VISIBILITY PROBLEMS

A guard occupies a point and can see in all directions to the walls of a polyhedron. What configurations of guards can see all points in a given polyhedron? What is the minimum number of guards?

Some Results:

- If a polyhedron is tetrahedralizable, then guards at vertices can cover it.
- The Schönhardt polyhedron is not tetrahedralizable, but vertex guards still cover it.
- Some polyhedra are impossible to cover with vertex guards
- Some polyhedra with n vertices require a number of guards proportional to $n\sqrt{n}$.

A **DISSECTION** of polygon P cuts P into a finite number of smaller polygons.

Polygons P and Q are **SCISSORS CONGRUENT** if P can be dissected into $P_1, P_2, \ldots, P_n$, which can be reassembled by translations and rotations to form Q.

slide

Note: cannot flip the pieces over (no reflections)

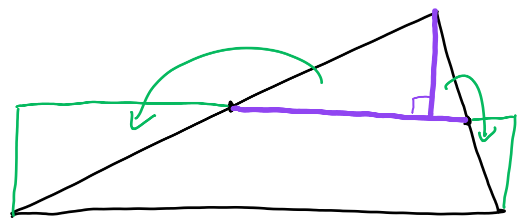
Question: If polygons P and Q have the same area, are they scissors congruent?

Scissors Congruence

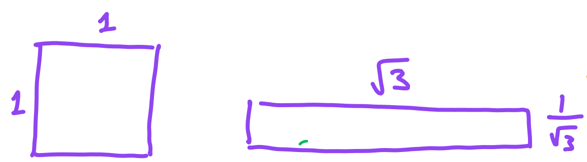
MATH 261 Computational Geometry

Question 1: Is every triangle is scissors congruent with some rectangle? *Find a proof or a counterexample.*

Yes!



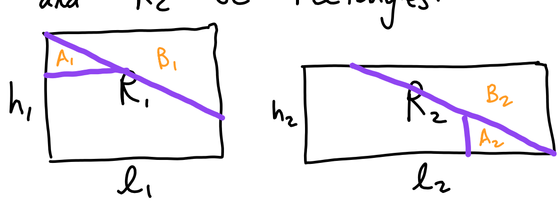
Question 2: Are every two rectangles of the same area are scissors congruent? *Find a proof or a counterexample.*



← "hard" example

Yes!

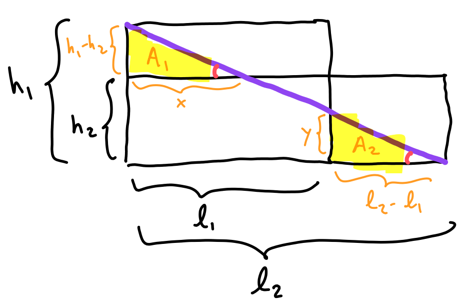
Let R_1 and R_2 be rectangles:



$A_1 = A_2$

$B_1 = B_2$

Assume: $h_2 < h_1 \le l_1 < l_2 < 2l_1$



similar triangles: $\frac{h_1 - h_2}{x} = \frac{h_1}{l_2} = \frac{y}{l_2 - l_1}$

observe: $\frac{h_1 - h_2}{l_2 - l_1} = \frac{h_1}{l_2}$ why? cross-multiply

$(h_1 - h_2)(l_2) = h_1(l_2 - l_1)$

$\cancel{h_1 l_2} - h_2 l_2 = \cancel{h_1 l_2} - h_1 l_1$

Areas of the rectangles are equal

So: $x = l_2 - l_1$

$y = h_1 - h_2$

Question 3: Suppose polygon P is scissors congruent to polygon Q, and Q is scissors congruent to polygon R. Is P scissors congruent to R? *Find a proof or a counterexample.*

Yes! Overlay both dissections onto Q

$$P \longleftrightarrow Q \longleftrightarrow R$$

Question 4: Let P and Q be polygons of the same area. Are P and Q scissors congruent? *Find a proof or a counterexample.*

Sketch of proof:

Triangulate P. Each triangle is scissors congruent to some rectangle. Rearranging the rectangles, P is scissors congruent to a rectangle R.

Also R is scissors congruent to Q.

So P and Q are scissors congruent.

Wallace-Bolyai-Gerwein Theorem

Does a similar theorem hold in 3D?
If two polyhedra have the same volume, are they scissors congruent?

Hilbert's third problem (1900)

David Hilbert (1900)

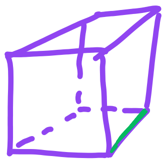
3. The Equality of the Volumes of Two Tetrahedra of Equal Bases and Equal Altitudes.

In two letters to Gerling, Gauss * expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, *i. e.*, in modern phraseology, upon the axiom of continuity (or upon the axiom of Archimedes). Gauss mentions in particular the theorem of Euclid, that triangular pyramids of equal altitudes are to each other as their bases. Now the analogous problem in the plane has been solved.† Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts. Nevertheless, it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility. This would be obtained, as soon as we succeeded in *specifying two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra.*‡

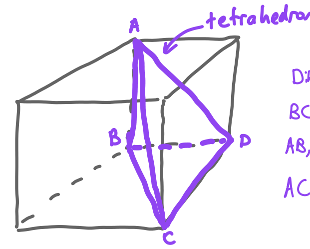
Hilbert wants an example of two tetrahedra of the same volume that are not scissors congruent.

Answered in 1903: by Dehn

[Problem also proposed by Kretowski in 1882, solved by Birkenmajer]



dihedral angles of a cube are all $90°$ or $\frac{\pi}{2}$ radians



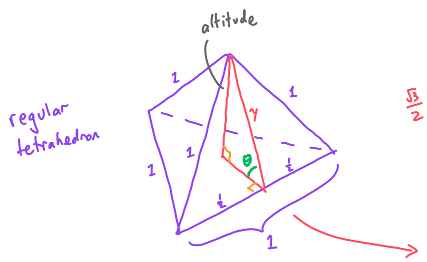
Dihedral angles:
BC, AD, BD: $\frac{\pi}{2}$
AB, CD: $\frac{\pi}{4}$
AC: $\frac{\pi}{3}$

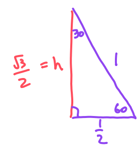
Scissors Congruence in 3D

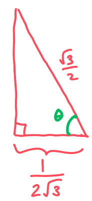
MATH 261 Computational Geometry

Let P be a polyhedron and let e be an edge of P. The **dihedral angle** at edge e is the interior angle between the two faces of P that intersect along edge e.

Question 1: What is the dihedral angle along any edge of a regular tetrahedron?



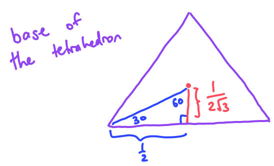


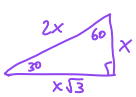


$$\cos\theta = \frac{\frac{1}{2\sqrt{3}}}{\frac{\sqrt{3}}{2}} = \frac{1}{\cancel{2}\sqrt{3}} \cdot \frac{\cancel{2}}{\sqrt{3}} = \frac{1}{3}$$

$$\boxed{\theta = \arccos\left(\frac{1}{3}\right)}$$

$\approx 70.53°$





NOTE: θ is not a rational multiple of π

Let $f : \mathbb{R} \to \mathbb{Q}$ be any function such that:

reals rationals

(a) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$,

(b) $f(qx) = qf(x)$ for all $q \in \mathbb{Q}$ and $x \in \mathbb{R}$,

(c) $f(\pi) = 0$.

d-function

Question 2: If θ is any rational multiple of π, then what can you say about $f(\theta)$?

$\theta = q\pi$ for some $q \in \mathbb{Q}$

Then $f(\theta) = f(q\pi) = q \cdot f(\pi) = q \cdot 0 = 0$

For any edge e of a polyhedron, let $\ell(e)$ be the length of e and $\phi(e)$ the dihedral angle at e. Call $\ell(e) \cdot f(\phi(e))$ the **mass** of edge e.

a d-function

The **Dehn invariant** of a polyhedron P is $D_f(P) = \sum_{\substack{\text{edge} \\ e \in P}} \ell(e) \cdot f(\phi(e))$.

Note dependence on the d-function f

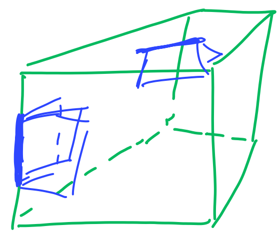
Theorem: Let f be any d-function. If polyhedron P is dissected into $P_1, P_2, \ldots, P_n$, then

$$D_f(P) = D_f(P_1) + D_f(P_2) + \cdots + D_f(P_n).$$

Proof: Let e be an edge in the dissection of P.
There are 3 ways that e can appear:

(1) e is contained in an edge of P

Let $\phi(e)$ be the dihedral angle in P,
and let $\phi_1(e), \phi_2(e), \ldots, \phi_k(e)$ be the dihedral angles in the dissection.



Then: $\underbrace{\ell(e)\left[f(\phi_1(e)) + \cdots + f(\phi_k(e))\right]}_{\text{dissection}} = \underbrace{\ell(e) f(\phi(e))}_{\text{original } P}$

So the mass of edge e is unchanged by dissection.

(2) e is contained in a face of P

Then the sum of the masses of edge e is $\ell(e) \cdot f(\pi) = 0$.

(3) e is contained in the interior of P

Again, $\ell(e) \cdot f(2\pi) = 0$.

Question 3: Let P and Q be polyhedra, and let f be any d-function. If $D_f(P) \neq D_f(Q)$, then can P and Q be scissors congruent?

No. If P and Q are scissors congruent,
then they must have $D_f(P) = D_f(Q)$

Question 4: How does your previous answer imply that a cube cannot be scissors congruent to a regular tetrahedron?

$D_f(\text{cube}) = 0$

$D_f(\text{reg. tetrahedron}) = 6\, \ell(e)\, f\left(\arccos\left(\tfrac{1}{3}\right)\right) \neq 0$

e.g. choose f such that
$f\left(\arccos\left(\tfrac{1}{3}\right)\right) = 1$