14 January 2025

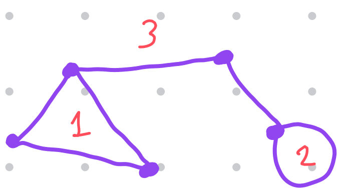
Euler's Formula

Let G be a ~~connected~~ connected planar graph with V vertices, E edges, and F faces (includes the unbounded outer face).

Then $V - E + F = 2$.

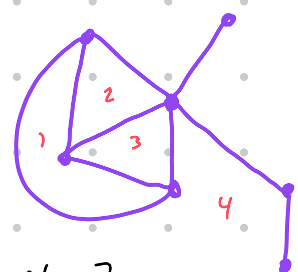
Examples:

faces



$V = 5$ vertices
$E = 6$ edges
$F = 3$ faces

$V - E + F = 5 - 6 + 3 = 2$



$V = 7$
$E = 9$
$F = 4$

$V - E + F = 7 - 9 + 4 = 11 - 9 = 2$

Proof that $V - E + F = 2$:

Induction on the number of edges.

Base Case: If $E = 0$, then G has one vertex and one face

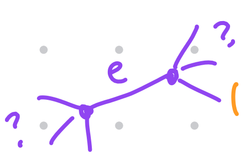
$V = 1$, $F = 1$.

Then: $V - E + F = 1 - 0 + 1 = 2$.

Induction: Assume Euler's formula holds for all graphs with $E - 1$ edges.

Let G have V vertices, E edges, and F faces. $(E \geq 1)$

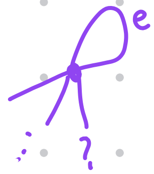
Choose any edge e of G.



(1) If e connects 2 different vertices: Contract e to form a new graph G' with $V-1$ vertices and $E-1$ edges.

By inductive hypothesis, $(V-1) - (E-1) + F = 2$.

So $V - E + F = 2$.



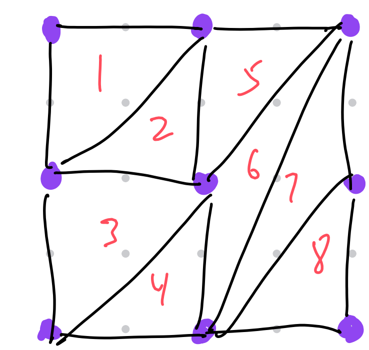
(2) If e is a loop, then delete e, to form a new graph G' with $E-1$ edges and $F-1$ faces.

By inductive hypothesis, $V-(E-1)+(F-1)=2$.

So $V-E+F=2$.

In either case, G satisfies $V-E+F=2$. ∎

QUESTION: How many triangles in a triangulation of a point set?



Planar graph G

$h = 8$ hull points

$k = 1$ interior point

Find t, the number of triangles

$$2k+h-2 = 2(1)+8-2 = 8 = t$$

How Many Triangles in a Triangulation?

MATH 261 Computational Geometry

Suppose point set S has h points on its convex hull and k points in its interior. (Assume the points of S are not all collinear.) Let planar graph G be a triangulation of S.

Use the following steps and Euler's formula to determine t, the number of triangles in G.

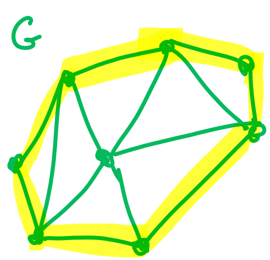
1. Express V, the number of vertices of G, in terms of h and k.

$V = h + k$

2. Express F, the number of faces of G, in terms of t.

$F = t + 1$

3. Express E, the number of edges of G, in terms of h, k, and t. To do this, count the edges of each triangular face, then add the edges of the unbounded exterior face.



t triangles have a total $3t$ edges (double-counts interior edges)

Convex hull has h edges

So: twice the number of edges is $2E = 3t + h$

thus, $\boxed{E = \frac{3t+h}{2}}$

4. Express Euler's formula in terms of h, k, and t. Then solve for t.

$$V - E + F = \underbrace{(h+k) - \frac{3t+h}{2} + (t+1) = 2}_{\text{solve for } t}$$

$$2h + 2k - 3t - h + 2t + 2 = 4$$

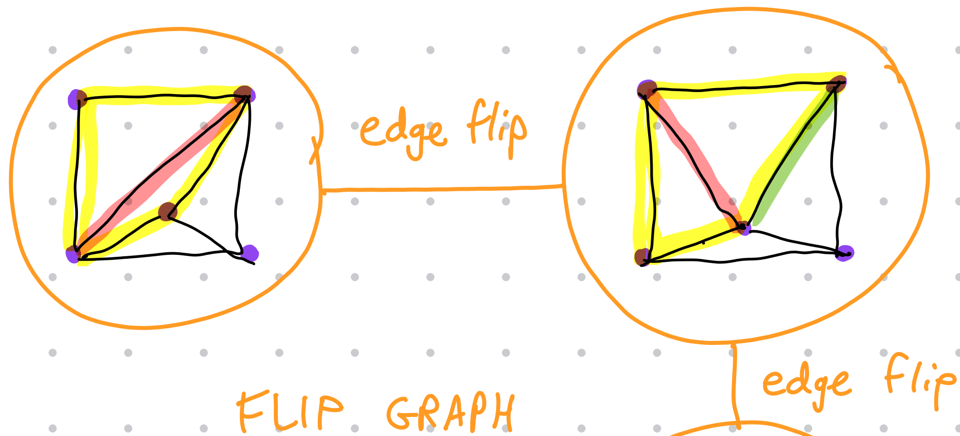
$$h - t + 2k = 2$$

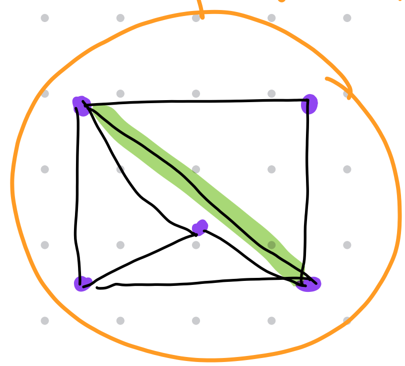
Bonus: How many edges are in G?

$\boxed{h + 2k - 2 = t}$ ← Number of triangles

Observation: Sometimes, two triangulations are <u>almost</u> the same, but differ by a single "edge flip."

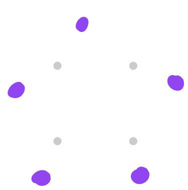
EXAMPLE:

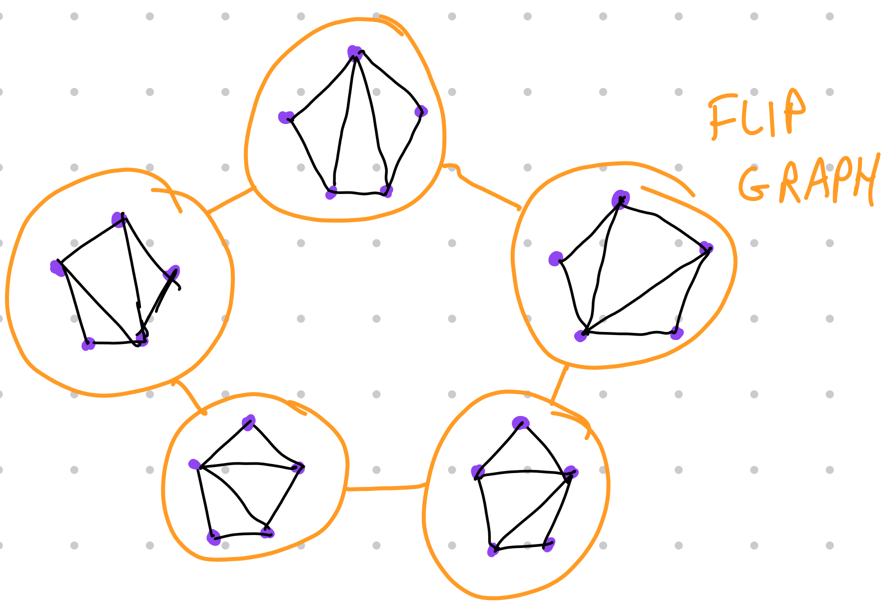




The **FLIP GRAPH** of a point set S is a graph whose vertices are the triangulations of S. Two vertices T_1 and T_2 are connected by an edge if one diagonal of T_1 can be flipped to obtain T_2.

EXAMPLE: Construct the flip graph for 5 points in convex position.





Flip Graphs

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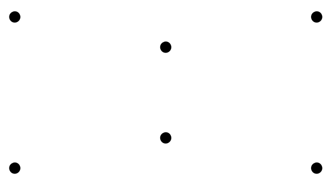
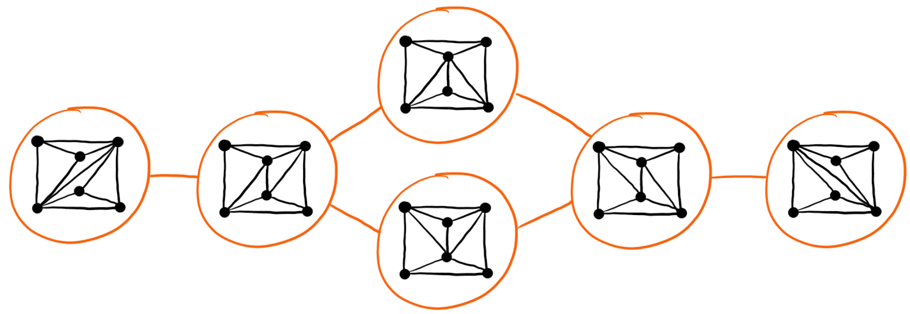
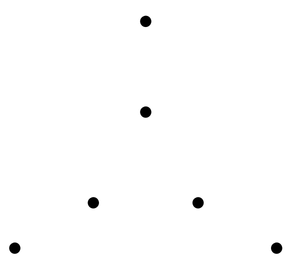
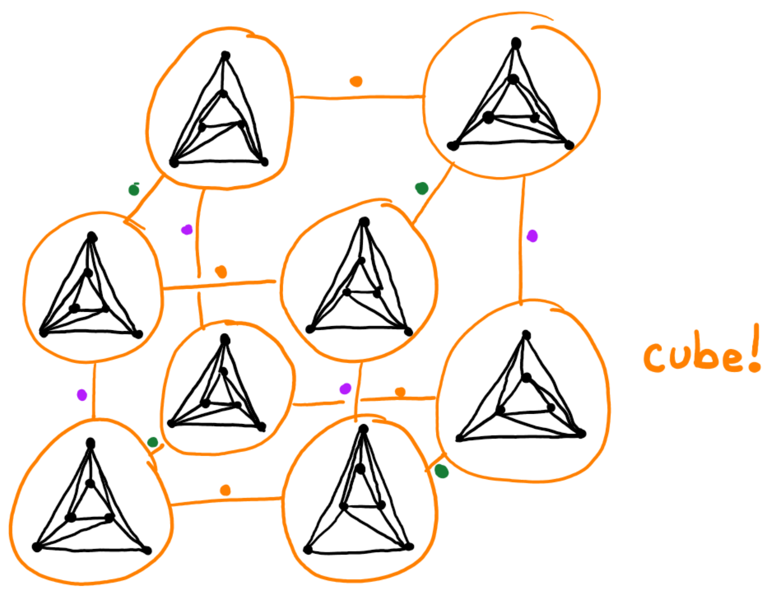
1. Construct the flip graph for the following point set.



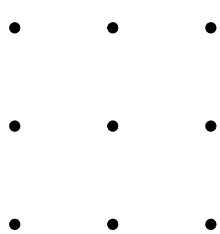


2. Construct the flip graph for the following point set.





3. Let S be the 3×3 grid of points, shown below. Find triangulations T_1 and T_2 of S such that the number of edge flips required to transform T_1 into T_2 is as large as possible. (This gives the *diameter* of the flip graph of S).



QUESTION: Is the flip graph connected?

Triangulations and Edge Flips

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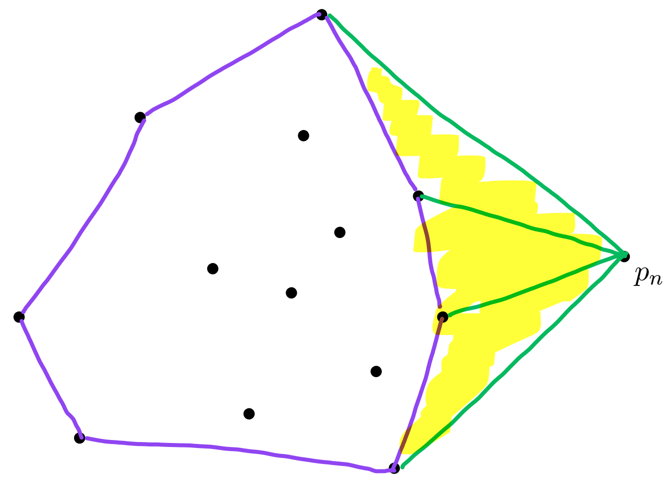
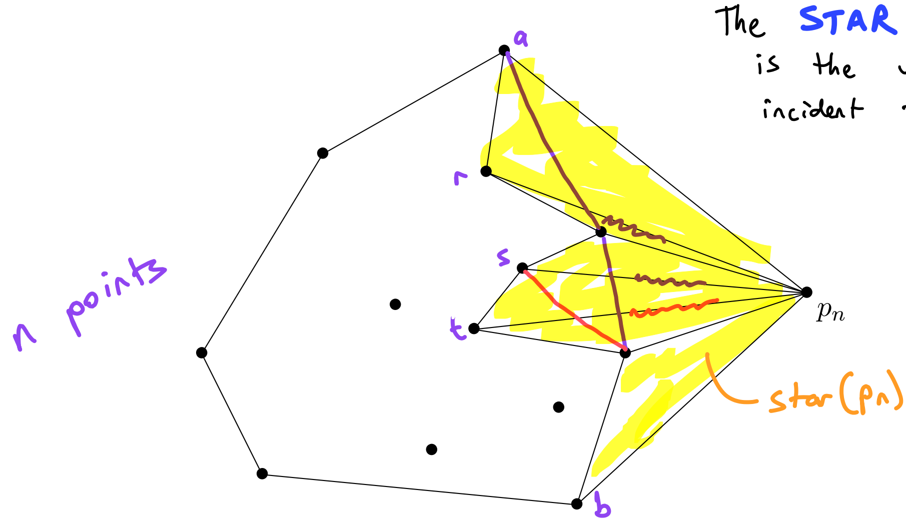
1. Consider the following set of points S. Label the points $p_1, p_2, \ldots, p_n$ from left to right. If the incremental algorithm is used to triangulate S, which triangles are incident to p_n? Draw all such triangles below.



2. Suppose a triangulation of S (same S as above) includes the edges shown below. Find a sequence of edge flips that transform these edges into the edges you drew above. That is, you want to transform the edges such that the triangles incident to p_n are exactly those produced by the incremental algorithm on S.

The STAR of a vertex is the union of all triangles incident to that vertex



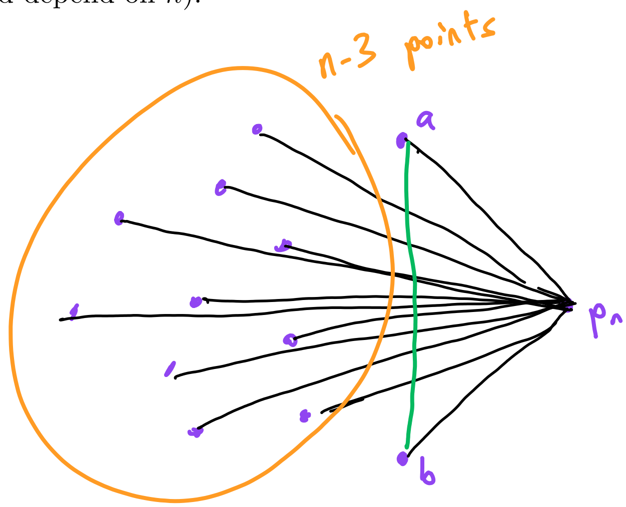
3. Generalize your observations from #1 and #2. Let S be *any* set of n points in the plane, and let p_n be the rightmost point of S. Given *any* triangulation of S, can you always find a sequence of edge flips that result in the triangles incident to p_n being exactly those produced by the incremental algorithm on S? If so, find an algorithm that achieves the edge flips. If not, give a counterexample.

Yes

n points, including p_n

4. In your algorithm for #3, what is the largest number of edge flips that might be required? (Your answer should depend on n).

$n-3$



5. What is the largest number of edge flips that might be required to transform some triangulation of S into the triangulation produced by the incremental algorithm? What does this imply about the diameter of the flip graph of S? (The *diameter* of the flip graph is the length of the longest path between any two nodes in the graph.)

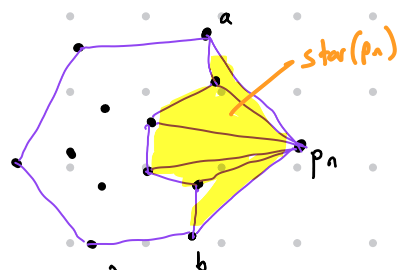
THEOREM: The flip graph of any point set in the plane is connected.

Proof: First, order points by x-coordinate.
Let T_* be the triangulation resulting from the incremental algorithm.
We will show (by induction on the number of points) that <u>any</u> triangulation can be transformed into T_* by edge flips.

base case: If $n=3$, then the only triangulation is a triangle. In particular, T_* is this triangle.

induction: Assume the flip graph is connected whenever the point set has fewer than n points.

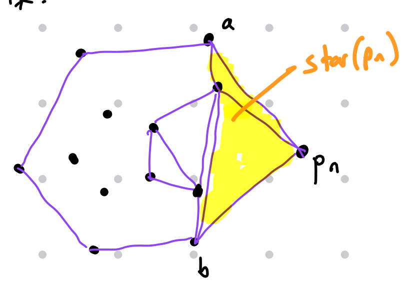
Let $S = \{p_1, p_2, .., p_n\}$ with p_n the rightmost point.
Let T be any triangulation of S.
Let a and b be the hull vertices adjacent to p_n.



In T, repeatedly flip edges connecting p_n to convex vertices in star(p_n).
(This process decreases degree(p_n), and thus must terminate.)

When the convex vertices in star(p_n) are only at p_n, a, and b, then star(p_n) consists of triangles in T_*.

Remove vertex p_n. The remainder of T can be converted to the triangulation of $\{p_1, p_n, \dots, p_{n-1}\}$ that results from the incremental algorithm (by induction hypothesis).



Thus, T_* can be obtained from T by edge flips. ∎

The **DIAMETER** of a graph is the length of the shortest path between the most distant nodes in the graph

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For a planar point set, n points any triangulation can be transformed into the incremental algorithm triangulation by how many edge flips?

at most $(n-3) + (n-4) + (n-5) + \cdots + 3 + 2 + 1 = \dfrac{(n-3)(n-2)}{2}$

$$\begin{array}{ccccccccccccc} & 1 & + & 2 & + & 3 & & n-5 & + & n-4 & + & n-3 & \\ \hline & (n-2) & + & (n-2) & + & (n-2) & + & (n-2) & + & (n-2) & + & (n-2) & = (n-3)(n-2) \end{array}$$

$$T_1 \xrightarrow{\frac{(n-3)(n-2)}{2}} \underset{\text{inc. alg.}}{T_*} \xrightarrow{\frac{(n-3)(n-2)}{2}} T_2$$

diameter of the flip graph is not greater than $(n-3)(n-2)$