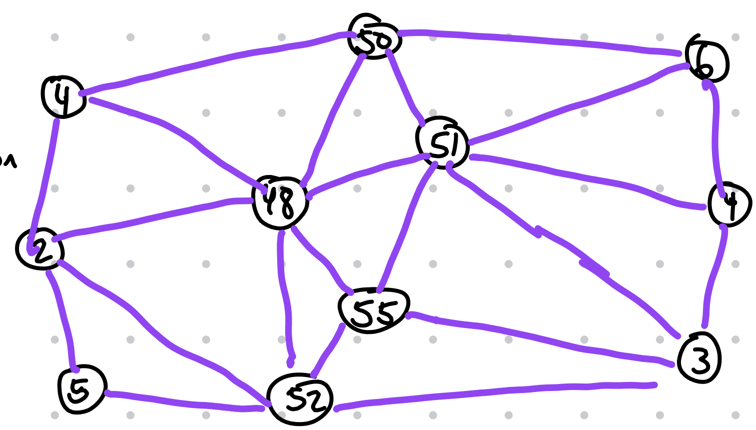
15 January 2025

Question: Which triangulation is best?

EXAMPLE: surface meshing, elevation modeling

What features of the triangulation make it the best?



Possible criteria for the "best" triangulation:

- Minimize total edge length
- Triangles as close to equilateral as possible
- Use angles close to 60 degrees — try to choose the largest smallest angle
- Edge flips: not allow situations where edge flips can lengthen edges
- Minimize largest degree of all vertices
- Connect hull points to closest vertices
- Avoid nearly-collinear points

LEXICOGRAPHICAL ORDER

EXAMPLE: math < matrix < tetrahedron < triangle

math < mathematics

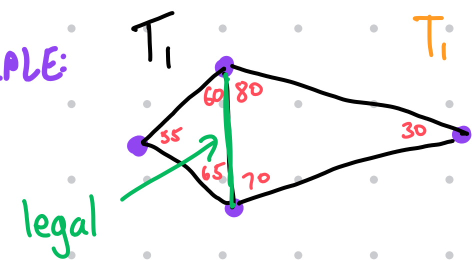
Given two sequences of numbers $A = (a_1, a_2, a_3, \ldots)$ and $B = (b_1, b_2, b_3, \ldots)$, we say $A < B$ if $a_i < b_i$ where i is the first index such that $a_i \neq b_i$.

EXAMPLE: $A = (10, 5, 80, 120)$ $\quad$ $B = (10, 7, 2, -1)$

$A < B$ since $a_1 = b_1$ and $a_2 < b_2$

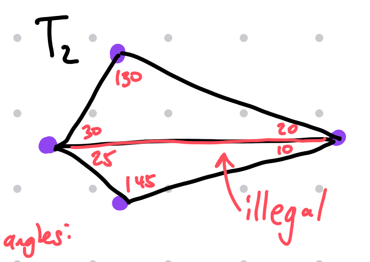
TRIANGULATIONS: For a given point set S, triangulation T_1 is "fatter" than triangulation T_2 if the sorted list of angles of T_1 is lexicographically larger than that of T_2.

EXAMPLE: T_1



T_1 is "fatter" than T_2

T_2



angles: $(30, 55, 60, 65, 70, 80)$

lexicographically larger

angles: $(10, 20, 25, 30, 130, 145)$

Of *all* triangulations of S, the "fattest" triangulation has the lexicographically largest angle sequence.

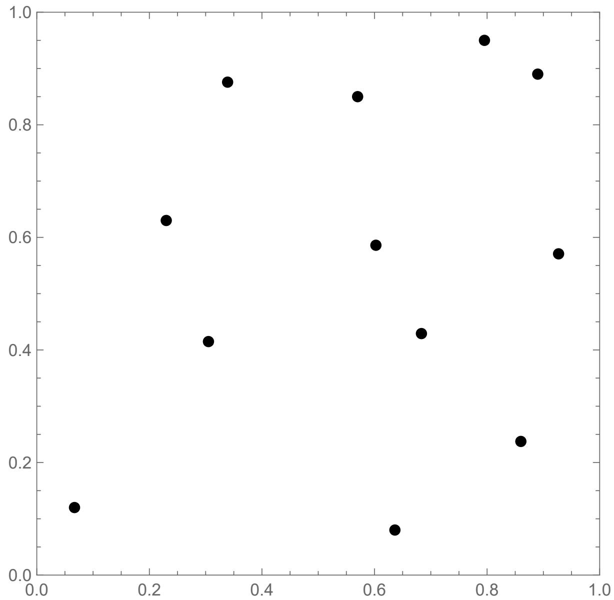
If an edge flip allows you to choose between two edges, the edge that results in the largest angle sequence is **LEGAL** and the other edge is **ILLEGAL**.

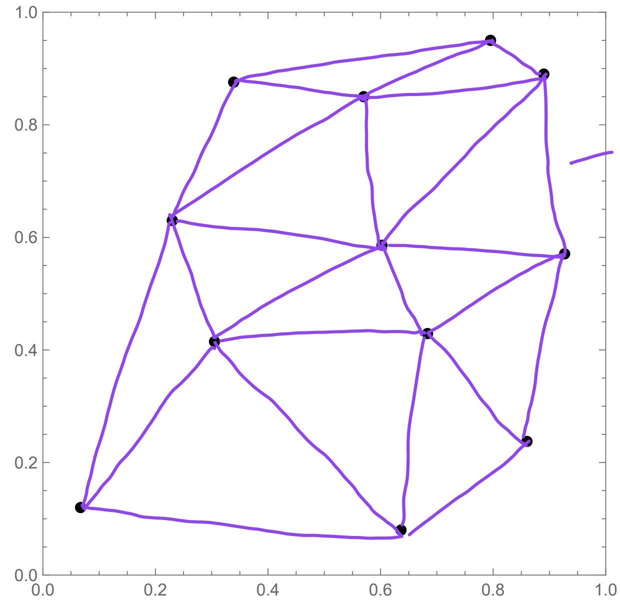
The “Best” Triangulation

MATH 261 Computational Geometry

Suppose that the “best” triangulation of a point set avoids long narrow triangles as much as possible. How would you make this mathematically precise?

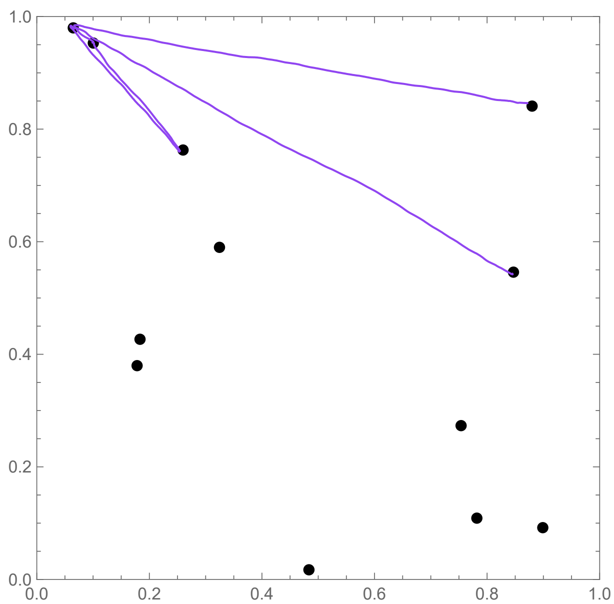
Make two (or more) triangulations of the following point set. Which one is “best”?

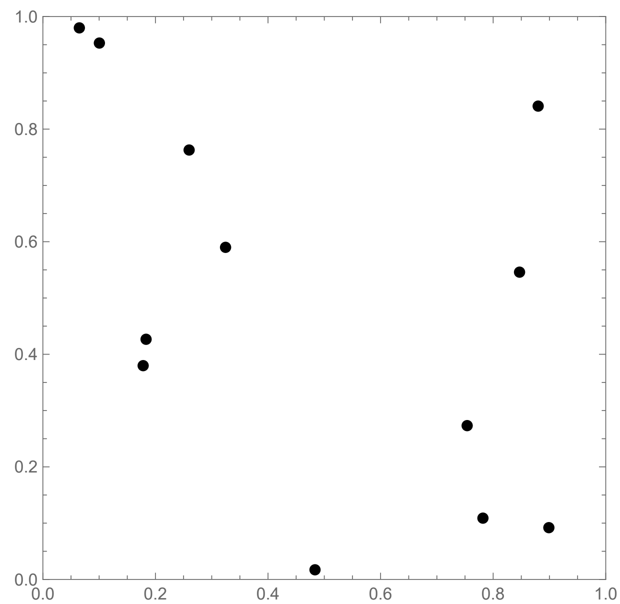




Delaunay triangulation.

Make two (or more) triangulations of the following point set. Which one is “best”?





How would you write a precise definition for the “best” triangulation of a point set?

"Legal" Edges

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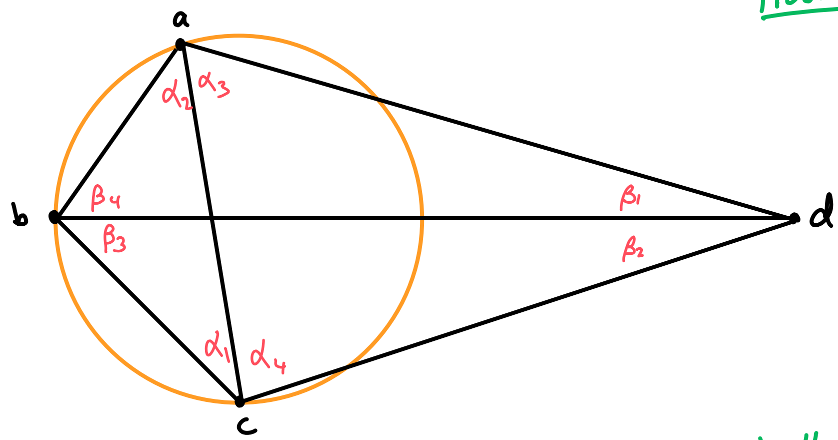
1. Justify the following: Let a, b, c, d be points in the plane, with a triangulation consisting of triangles Δabc and Δacd. Then edge $\overline{ac}$ is a legal edge if and only if d is outside of the circumcircle of Δabc.



Proof $\Leftarrow$:

If d is outside the circle, then $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$.

Similarly, $\alpha_3 > \beta_3$ and $\alpha_4 > \beta_4$.

Assume $\alpha_1 < \alpha_2$.

Triangulation with edge $\overline{ac}$ has angle sequence:

$\alpha_1, \alpha_2, \ldots$

lexicographically larger than

Triangulation with edge $\overline{bd}$ has angle sequence

$\beta_1, \ldots$

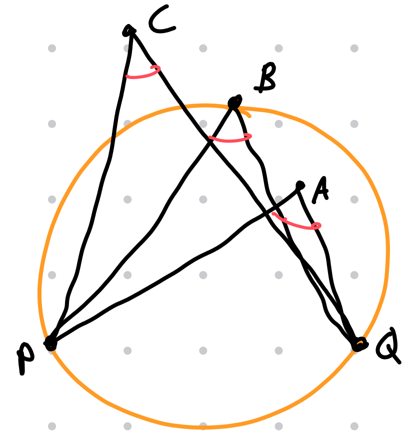
So edge $\overline{ac}$ is legal.

2. Justify the following: Let a and b be points in a planar point set S. If there exists a circle passing through a and b that contains no other points of S in its interior, then edge $\overline{ab}$ is a legal edge.

THALES'S THEOREM:

Let P, Q, B be points on a circle, A inside the circle, and C outside.

Then: $\measuredangle PAQ > \measuredangle PBQ > \measuredangle PCQ$.



The **DELAUNAY TRIANGULATION** is the triangulation that has no illegal edges.

Also, the Delaunay triangulation has the property that no point of S lies within the circumcircle of any triangle in the triangulation.

DELAUNAY TRIANGULATION: the triangulation consisting of no illegal edges (all edges are hull edges or legal edges).

This is also the "fattest" triangulation by lexicographical order of angle sequences.

Empty Circle Property: Triangulation T is the Delaunay triangulation if and only if no point of S lies in the circumcircle of any triangle in T.

Also, if $a, b \in S$ and there is some circle through a and b with no other points of S in its interior, then edge ab is in the Delaunay triangulation.

The Delaunay Triangulation

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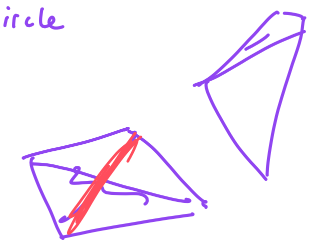
1. Let S be a planar point set, and start with any triangulation T of S. Repeat the following process: If T has an illegal edge, flip it to remove it. Will this lead to the Delaunay triangulation of S? Why or why not?

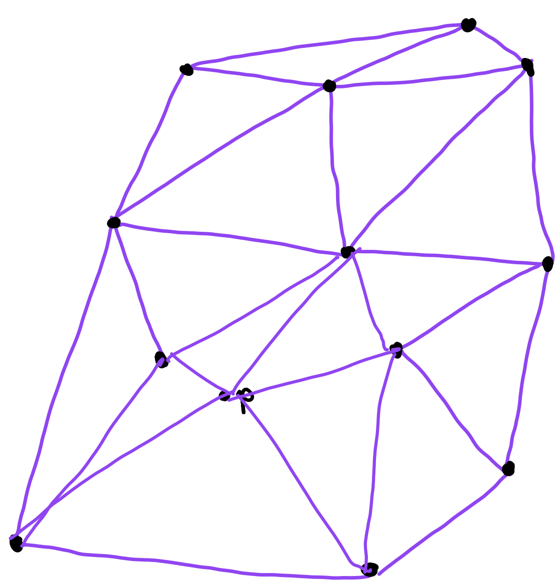
Yes.

Flip graph is connected.

Edge flips of illegal edges to legal edges transform triangles without the empty circle property into triangles with the empty circle property.



2. Suppose you have a Delaunay triangulation of a set of points S. If you add a new point p, describe an algorithm for updating the triangulation to obtain the Delaunay triangulation of $S \cup \{p\}$.



(1981)

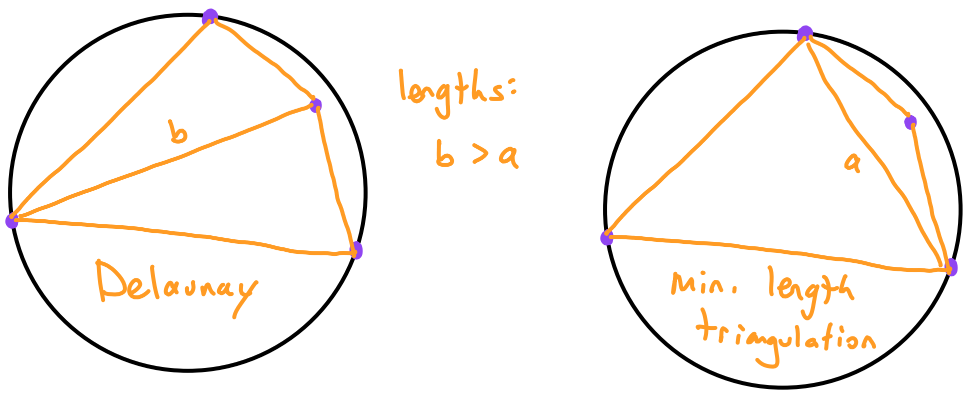
Bowyer–Watson Algorithm

Start with a triangle.

Add points one at a time; for each, remove any triangle (edge?) whose circumcircle contains the new point. Then re-triangulate the hole by adding edges from the new point.

MINIMUM LENGTH TRIANGULATION of a point set S is the triangulation with the smallest total edge length.

3. Does the Delaunay triangulation minimize the total length of all edges in a triangulation? Either explain why or find a counterexample.



4. Consider the following *greedy algorithm* for constructing a triangulation of a set of n points: Compute all $\binom{n}{2}$ distances between pairs of points. Consider all possible edges from shortest to longest, and decide whether to include each edge in the triangulation. If the edge does not cross any previously included edge, then include it in the triangulation.

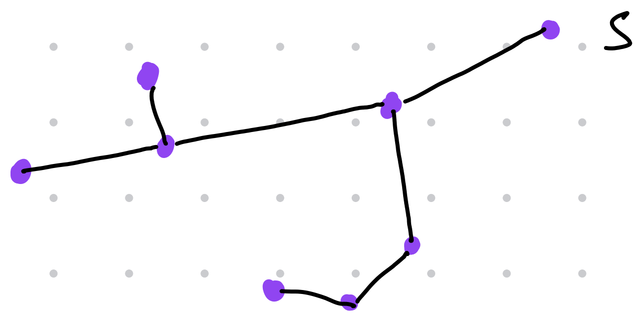
Does this algorithm produce the Delaunay triangulation? Does it produce the min-~~weight~~ length triangulation?

No – counterexample in #3

No – see page 89 in the text

MINIMUM SPANNING TREE:

For a point set S, the minimum spanning tree is the tree (graph with no loops) that connects all points in S with smallest total length.



THEOREM: For a point set S, the minimum spaning tree of S is contained in the Delaunay triangulation of S.