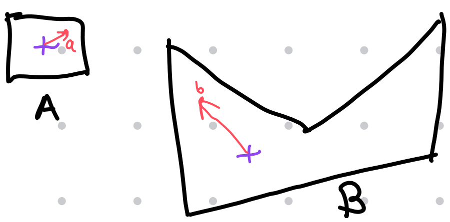
22 January 2025

MINKOWSKI SUM

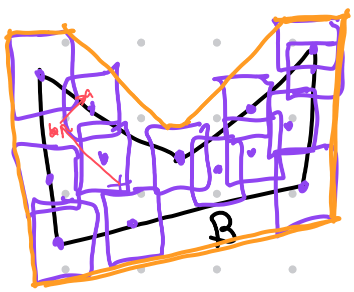
Suppose we have two polygons, each with a marked point.

origin, a base point



For each point $x \in B$, position a copy of A such that its origin is at x.

Take the union of all such copies of A.



This union is the **MINKOWSKI SUM** of A and B, denoted

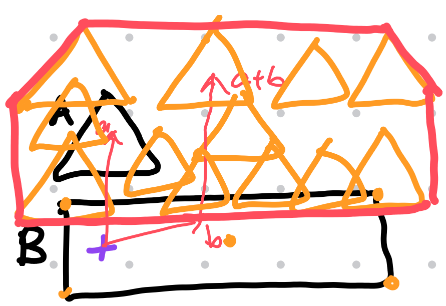
$$A \oplus B.$$

LaTeX: \oplus

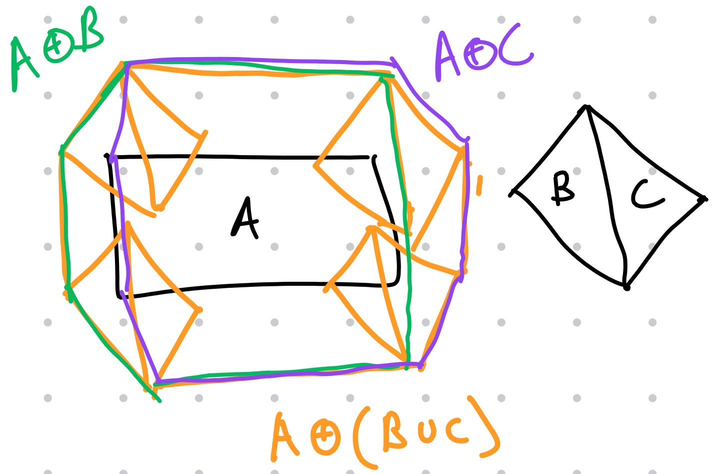
Formally, $A \oplus B = \{ x + y \mid x \in A \text{ and } y \in B \}$

where $x+y$ is the vector sum of the two points.

EXAMPLE:



$A \oplus B$

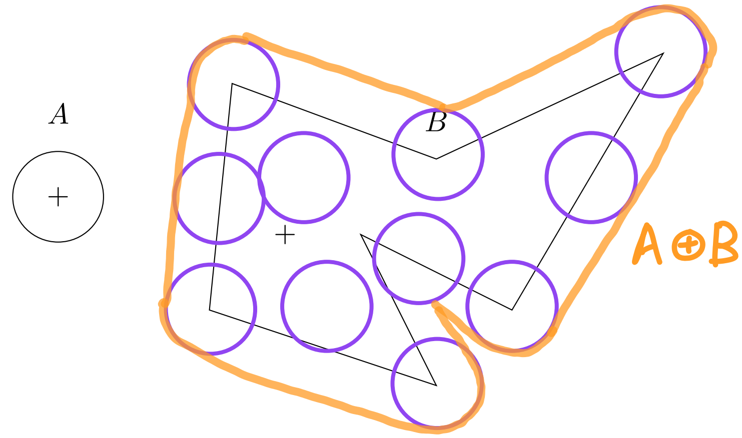

A ⊕ B
A ⊕ C
A
B
C
A ⊕ (B ∪ C)

Minkowski Sums

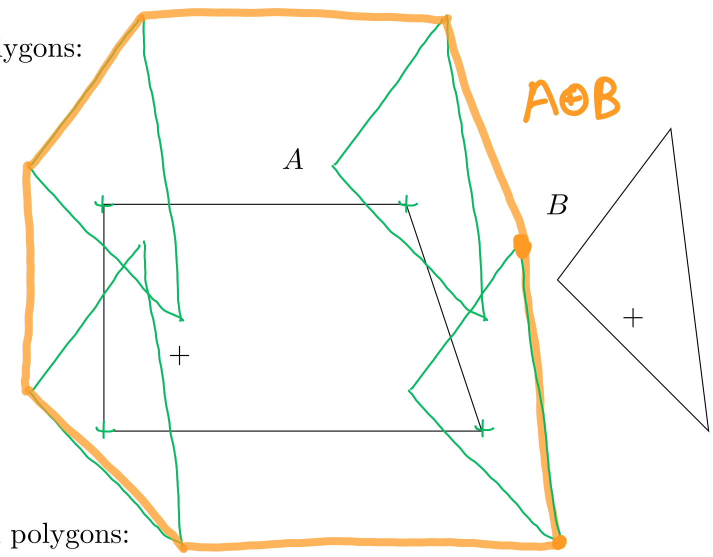
MATH 261 Computational Geometry

For each pair of point sets A and B, construct the Minkowski sum $A \oplus B$.

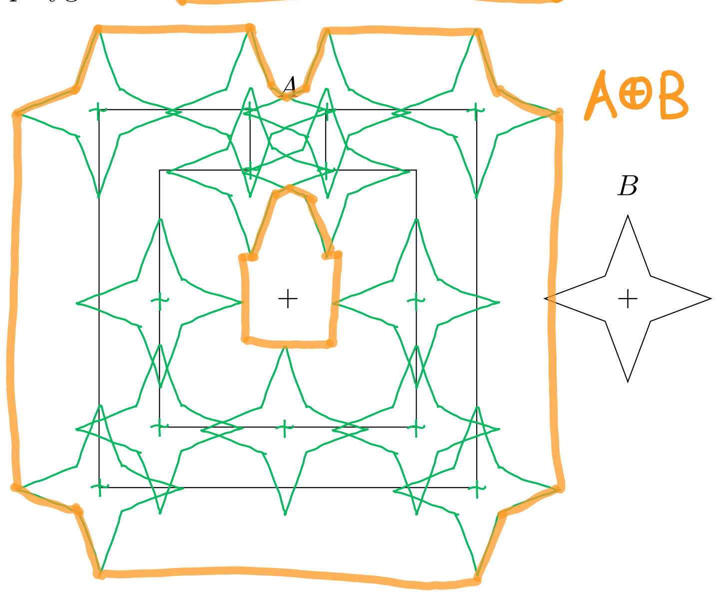
1. A disk and a polygon:



2. Two convex polygons:



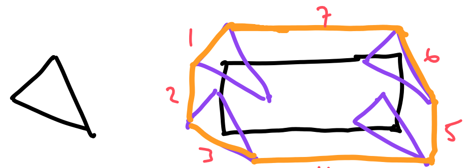
3. Two nonconvex polygons:



4. Let A and B be convex polygons with m and n vertices, respectively. How many vertices does $A \oplus B$ have?

also convex

$A \oplus B$ has at most $m+n$ vertices, and fewer if A and B have parallel edges



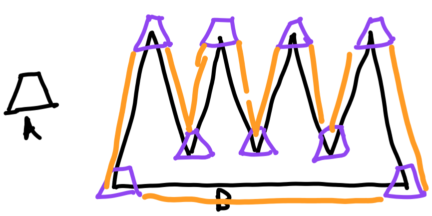
5. How are $A \oplus B$ and $A \oplus C$ related to $A \oplus (B \cup C)$?

$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$$

y is in B or in C

$$A \oplus (B \cup C) = \{x + y \mid x \in A, y \in B \cup C\}$$
$$= \{x+y \mid x \in A, y \in B\} \cup \{x+y \mid x \in A, y \in C\}$$
$$= (A \oplus B) \cup (A \oplus C)$$

6. Let A be a convex polygon with m vertices and B a nonconvex polygon with n vertices. What can you say about the number of vertices of $A \oplus B$?



Theorem: number of vertices of $A \oplus B$ is $O(mn)$

Proof involves triangulating B and applying #5

7. Let A and B be nonconvex polygons with m and n vertices, respectively. What can you say about the number of vertices of $A \oplus B$?

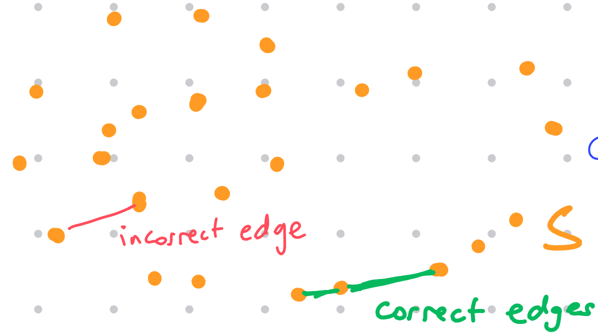
$A \oplus B$ has $O(m^2 n^2)$ vertices

8. Describe an algorithm for computing the Minkowski sum of convex polygons A and B. How would you adapt your algorithm for nonconvex polygons?

Consider all vector sums of vertices of A with vertices of B. Then take convex hull. $O(mn)$

PROBLEM: Suppose we have a finite set of points S sampled from an unknown curve C.
How can we approximate C from S?

EXAMPLE:



A **correct polygonal reconstruction** P of curve C connect points p and q if and only if p and q are consecutive sample points along C.

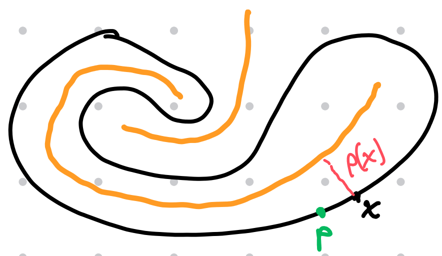
CRUST algorithm is a famous algorithm for solving this problem with provably correct bounds.

Input: Point set S (sampled from some simple closed curve)

1. Compute $\mathrm{Vor}(S)$, and let V be the set of Voronoi vertices.
2. Compute Delaunay triangulation $\mathrm{Del}(S \cup V)$.
3. Curve P is composed of the edges in $\mathrm{Del}(S \cup V)$ with both endpoints in S.

Provable Correctness of CRUST Algorithm

Let C be a planar closed curve, and let $x \in C$.
The **LOCAL FEATURE SIZE** $\rho(x)$ is the shortest distance from x to the medial axis of C.
rho



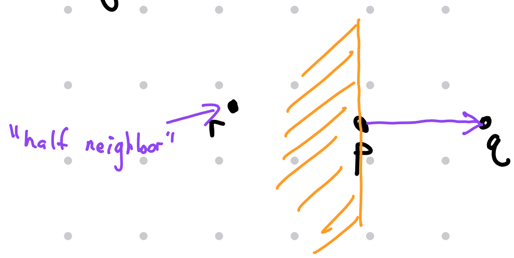
epsilon

Let $0 < \varepsilon < 1$. A set of points S sampled from C is **ε-SAMPLE** if each point $x \in C$ has a point $p \in S$ such that $|x - p| \leq \varepsilon \cdot \rho(x)$.

THEOREM: The CRUST algorithm outputs the correct polygonal reconstruction whenever S is an ε-sample with $\varepsilon \leq \frac{1}{5}$.

OTHER ALGORITHMS:

- Nearest Neighbor NN-CRUST: provably correct if $\varepsilon < \frac{1}{3}$



q is closest sample point to p
Include edge $\overline{pq}$ in the reconstruction.

include edge $\overline{pr}$

OPEN QUESTION
$0.66 \leq \varepsilon \leq 0.72$?

- Related algorithms: $\varepsilon < \frac{1}{2}$
- Compatible CRUST: provably correct for $\varepsilon < 0.66$
- Håvard Bjerkevik showed that correct reconstruction is not possible for $\varepsilon = 0.72$.

CRUST Algorithm

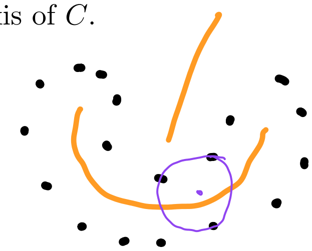
MATH 261 Computational Geometry

1. Suppose S is a sample of points from a curve C. Assuming the sample is sufficiently dense, justify the following statements:

(a) The Voronoi vertices of $\mathrm{Vor}(S)$ lie near the medial axis of C.

Voronoi vertices are equidistant from 3 points of S

Medial axis is equidistant from 2 closest points of S



not part of the reconstructed curve

correct Del(S)

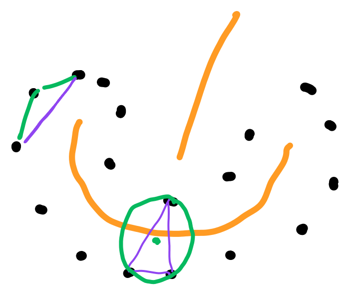
(b) Any circumscribing disk of an incorrect edge of the Delaunay triangulation $\mathrm{Del}(S)$ (an edge between two sample points that are not consecutive on C) crosses the medial axis of C.

Internal diagonals cross the medial axis.

INTUITION:

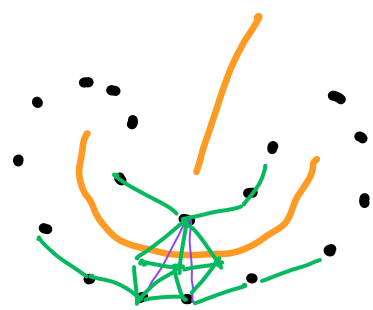
correct edges are "far" from the medial axis

incorrect edges are "close" to the medial axis



(c) Let V be the set of Voronoi vertices of $\mathrm{Vor}(S)$. An incorrect edge of $\mathrm{Del}(S)$ cannot also appear in the Delaunay triangulation $\mathrm{Del}(S \cup V)$.

Edges in Del(S∪V) will include the voronoi vertices of Vor(S), so triangles with incorrect edges of Del(S) won't be "fat" triangles.



(d) Each correct edge of $\mathrm{Del}(S)$ also appears in $\mathrm{Del}(S \cup V)$.

If sample points are close enough together, then correct edges are short enough that their circumscribing disks don't contain a point of V.

2. Suppose S is a sample of points from a curve C. Justify the following statements related to the NN-CRUST algorithm:

(a) Let $p \in S$ be any sample point and q its nearest neighbor. If the sample is sufficiently dense, then edge pq is correct.

(b) Let $p \in S$ be any sample point and q its half neighbor. If the sample is sufficiently dense, then edge pq is correct.