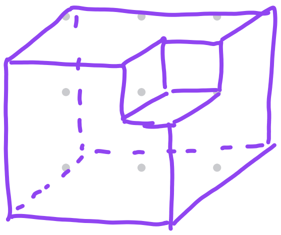
24 January 2025

Which of the following are polyhedra?

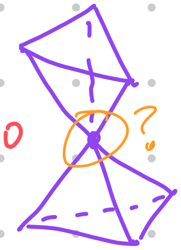
(a)

yes ✓



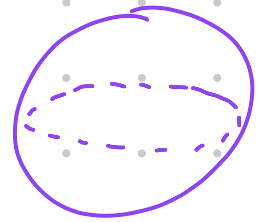
(b)

NO ?



(c)

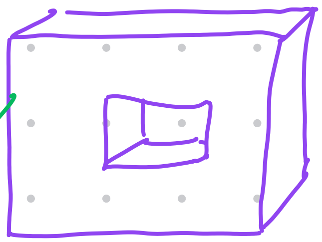
NO



(d)

yes? ✓

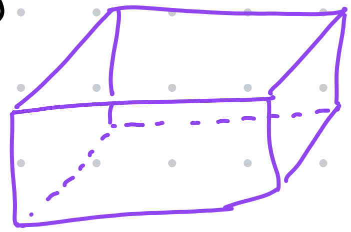
? no.



(e)

NO

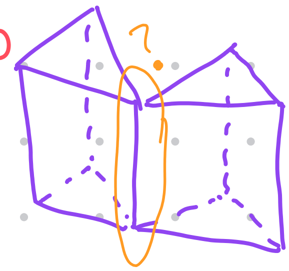
?



(f)

NO

?

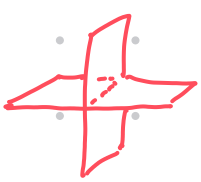


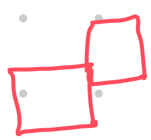
DEFINITION OF POLYHEDRON:

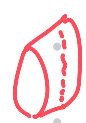
A polyhedron is a bounded region of space whose boundary is composed of a finite set of polygonal faces satisfying the following criteria:

1. Intersection Condition: any two faces intersect only at a single vertex or along a single edge

NOT ALLOWED:

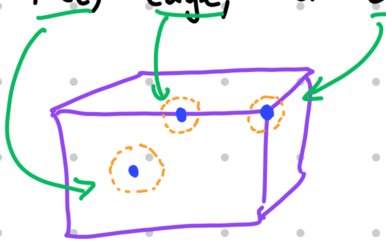




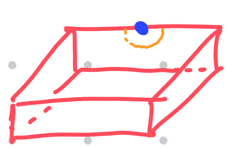


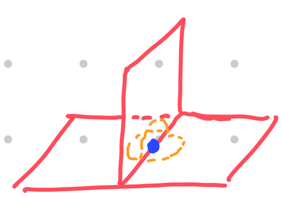
two faces that share two edges

2. Local Topology: Any small patch of a polyhedron looks like a face, edge, or corner.



Not allowed:

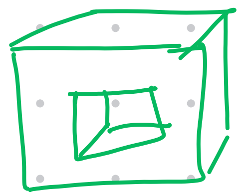




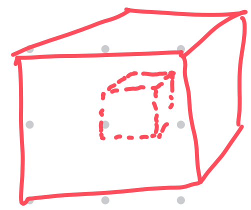
(Every point on the surface of a polyhedron has an open neighborhood homeomorphic to an open disk.)

← continuous bijection with a continuous inverse

3. Global Topology: Surface of the polyhedron is connected.



tunnels OK



cavities not OK

Polyhedral Investigation

MATH 261 Computational Geometry

For some of your favorite polyhedra, compute

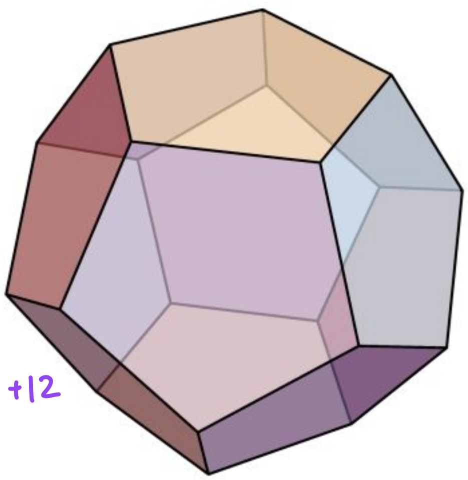
$$V - E + F$$

where V is the number of vertices, E is the number of edges, and F is the number of faces.

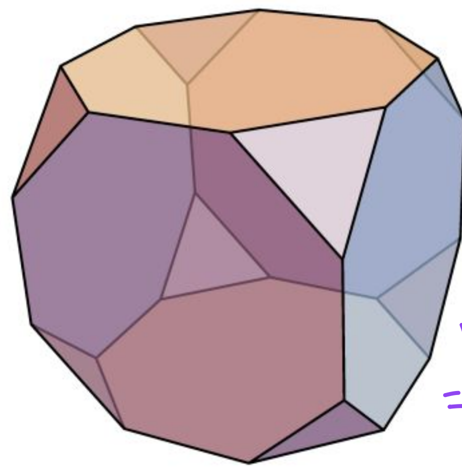
$V = 20$

$E = 30$

$F = 12$



$V - E + F = 20 - 30 + 12$
$= 2$

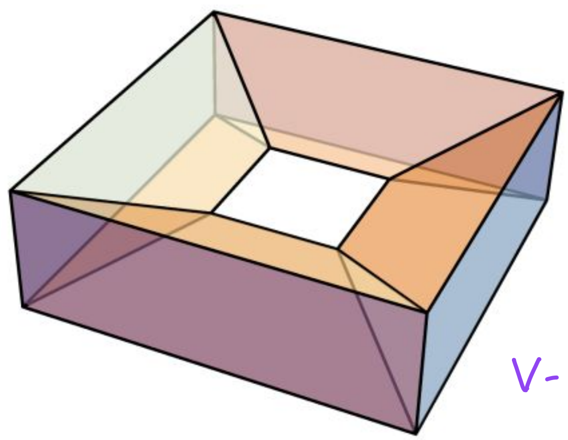


$V = 24$

$E = 36$

$F = 14$

$V - E + F$
$= 24 - 36 + 14$
$= 2$

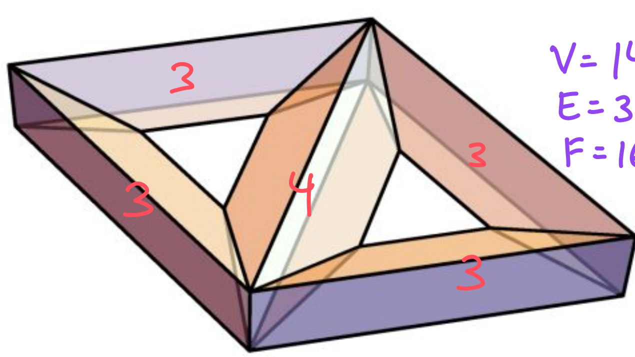


$V = 12$

$E = 24$

$F = 12$

$V - E + F = 12 - 24 + 12$
$= 0$



$V = 14$

$E = 32$

$F = 16$

$V - E + F$
$= 14 - 32 + 16$
$= -2$

V-E+F patterns

If polyhedron P has no holes/tunnels, then V-E+F = 2.
That is, $\chi(P) = 2$

For each hole/tunnel, V-E+F decreases by 2.

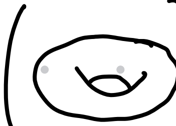
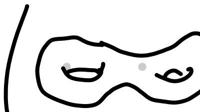
For any polyhedron P, the quantity V-E+F is called the **EULER CHARACTERISTIC** of P, denoted $\chi(P)$.
chi

Holes/Tunnels:

The number of holes/tunnels is called the **GENUS** of the polyhedron.

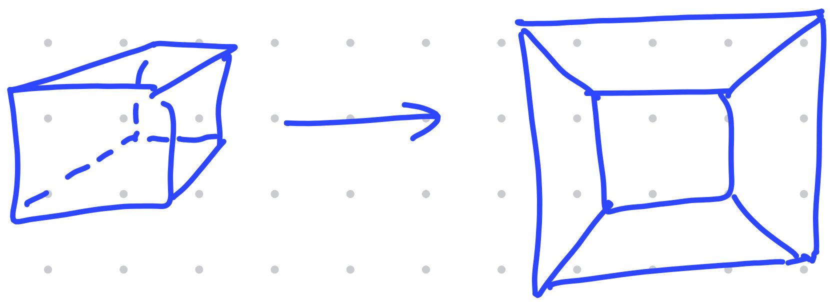
> **THEOREM:** For any polyhedron P, $\chi(P) = 2 - 2g$, where g is the genus of P.

This also generalizes to smooth surfaces:

$\chi($  $) = 2$, $\chi($  $) = 0$, $\chi($  $) = -2$

genus 0 genus 1 genus 2

$\chi = 2 - 2g$



1. Let P be a polyhedron of genus zero. If every face of P is either a pentagon or a hexagon, and if the degree of each vertex is 3, then how many faces are pentagons?

Suppose there are n pentagons and m hexagons.

Then: $F = m + n$

Total degree: $\underbrace{2E} = \underbrace{3V = 5n + 6m}$

$E = \frac{5n+6m}{2}$ $\qquad$ $V = \frac{5n+6m}{3}$

Now use Euler's formula: $V - E + F = 2$

$$2\cancel{6}\frac{(5n+6m)}{\cancel{3}} - {}^{3}\cancel{6}\frac{(5n+6m)}{\cancel{2}} + 6(n+m) = 2 \cdot 6$$

$$10n + \cancel{12m} - (15n + \cancel{18m}) + 6n + \cancel{6m} = 12$$

$$\boxed{n = 12}$$ 12 pentagons!

2. Let P be a polyhedron of genus zero. If every face of P is a traingle and the degree of each vertex is either 5 or 6, then how many vertices have degree 5?