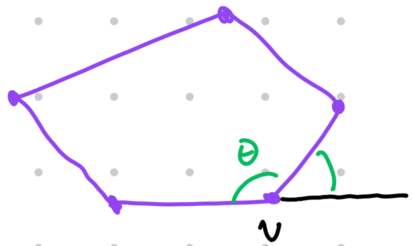
27 January 2025

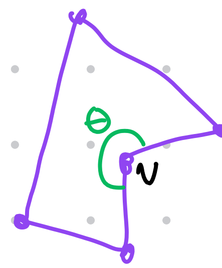
CURVATURE OF POLYGONS

ANGULAR DEFECT measures how much a polygon bends at each vertex

"curvature" is concentrated at the vertices



angular defect
$K(v) = \pi - \theta$



$K(v) = \pi - \theta < 0$

convex vertices: $K(v) > 0$
reflex vertices: $K(v) < 0$

Add up angular defect at all vertices of a polygon:

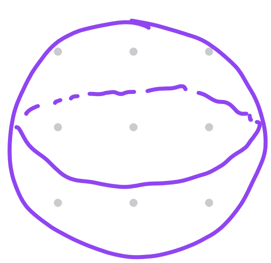
$$\sum_{v \in P} K(v) = 2\pi$$

<u>Recall</u>: sum of interior angles of an n-gon is $(n-2)\pi$

CURVATURE OF POLYHEDRA

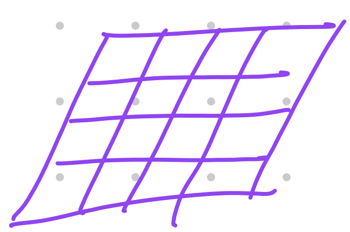
3D Curvature may be:

positive



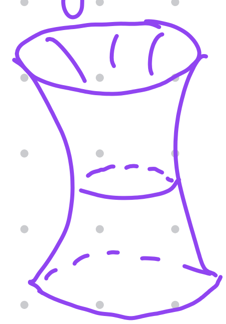
sphere

zero



plane

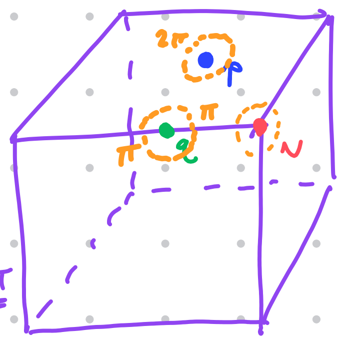
negative



hyperboloid

For polyhedra: The **CURVATURE** $K(p)$ at a point p on a polyhedron is 2π minus the sum of the face angles at p.

EXAMPLES:



Point on a face: $K(p) = 2\pi - 2\pi = 0$

Point on an edge: $K(q) = 2\pi - (\pi + \pi) = 0$

Point on a vertex:

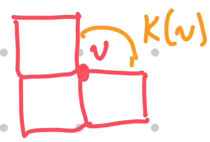
$$K(v) = 2\pi - \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi}{2}$$

Total curvature of cube:

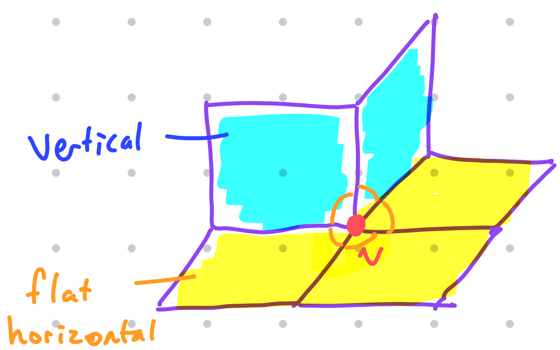
$$\sum_{\text{vertices } v} K(v) = 8\left(\frac{\pi}{2}\right) = 4\pi$$

NOTE: curvature is zero except at vertices

note: "angle defect" at a vertex



non-convex vertex:



curvature at v:

$$K(v) = 2\pi - \left(5 \cdot \frac{\pi}{2}\right) = -\frac{\pi}{2}$$

negative curvature!

Polyhedral Curvature

MATH 261 Computational Geometry

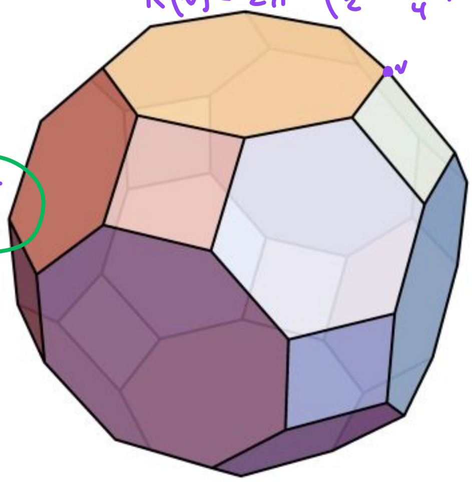
Compute the Gaussian curvature at each vertex of the polyhedron. Then add up the Gaussian curvature over all vertices of the polyhedron.

$K(v) = 2\pi - \left(\frac{\pi}{2} + \frac{3\pi}{4} + \frac{2\pi}{3}\right) = \frac{\pi}{12}$

Total Curvature:

$48\left(\frac{\pi}{12}\right) = 4\pi$

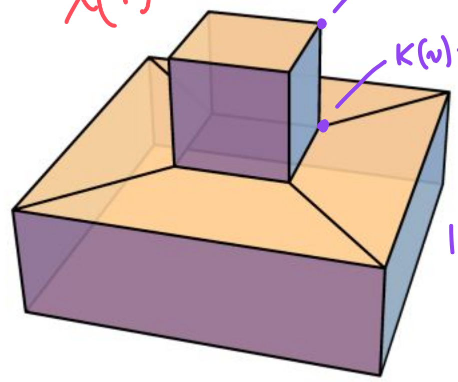
$\chi(P) = 2$



$\chi(P) = 2$

$K(v) = \frac{\pi}{2}$

$K(v) = \frac{-\pi}{2}$



Total:

$12\left(\frac{\pi}{2}\right) + 4\left(\frac{-\pi}{2}\right) = 4\pi$

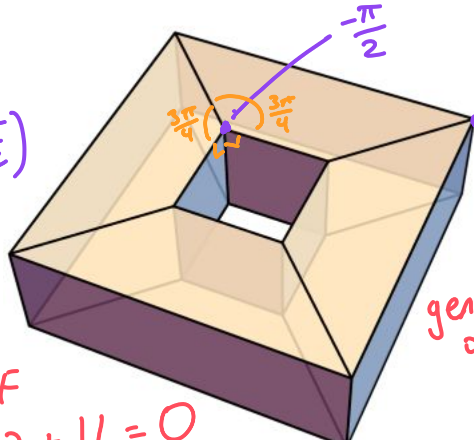
$-\frac{\pi}{2}$

$\frac{3\pi}{4}$ $\frac{3\pi}{4}$

$\frac{\pi}{2}$

Total:

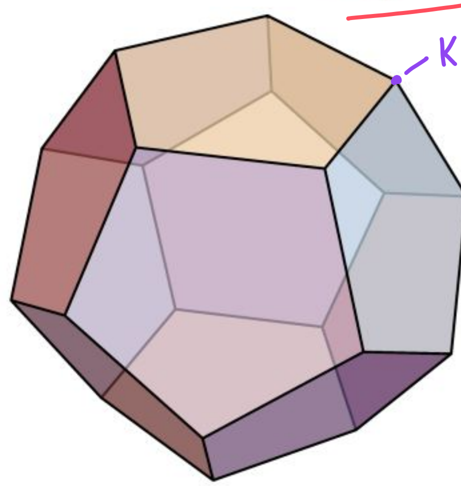
$8\left(\frac{\pi}{2}\right) + 8\left(\frac{-\pi}{2}\right) = 0$



genus one

$\chi(P) = V - E + F = 16 - 32 + 16 = 0 = 2 - 2g$

$K(v) = 2\pi - 3\left(\frac{3\pi}{5}\right) = \frac{\pi}{5}$



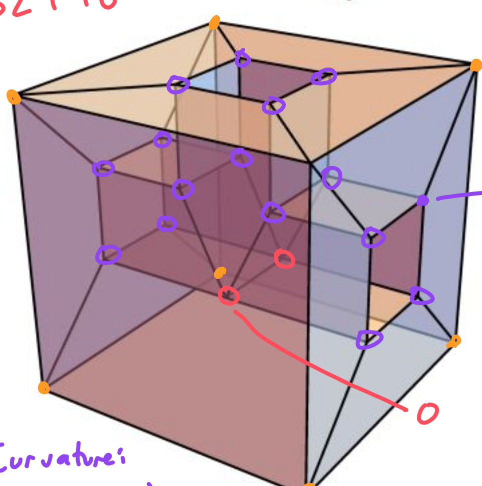
Total:

$20\left(\frac{\pi}{5}\right) = 4\pi$

$\chi(P) = 2$

$\frac{\pi}{2}$

$-\frac{\pi}{2}$



0

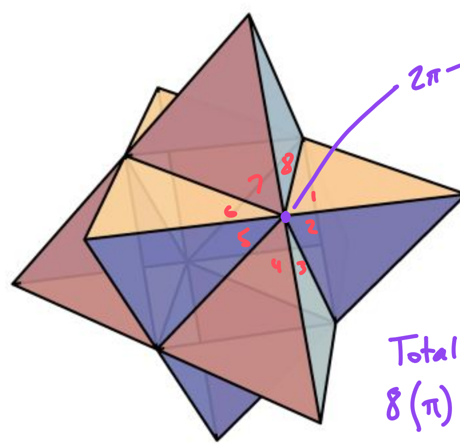
Total Curvature:

$8\left(\frac{\pi}{2}\right) + 16\left(\frac{-\pi}{2}\right) = 4\pi - 8\pi = -4\pi$

$\chi(P) = -2$ genus $= 2$

$2\pi - \left(8 \cdot \frac{\pi}{3}\right) = \frac{-2\pi}{3}$

$2\pi - \left(3 \cdot \frac{\pi}{3}\right) = \pi$



Total Curvature:

$8(\pi) + \cancel{6}\,2\left(\frac{-2\pi}{\cancel{3}}\right) = 4\pi$

$\chi(P) = 2$

POLYHEDRAL GAUSS-BONNET THEOREM

For any polyhedron P, the total curvature is 2π times the Euler Characteristic:

$$\underbrace{\sum_{v \in P}}_{\text{vertices}} \underbrace{K(v)}_{\text{curvature}} = 2\pi \underbrace{\chi(P)}_{\text{Euler characteristic}}$$

GEOMETRIC

(angles)

COMBINATORIAL

(counting)

For each of the following polyhedra, let R be the region above the curve indicated by arrows. Verify the formula:

$$\sum_{v\in R\setminus\partial R} K(v) + \sum_{v\in\partial R} K(v) = 2\pi\chi(R).$$