28 January 2025

QUESTION: If two polyhedra have congruent faces arranged similary around each vertex, are they congruent?

combinatorially equivalent

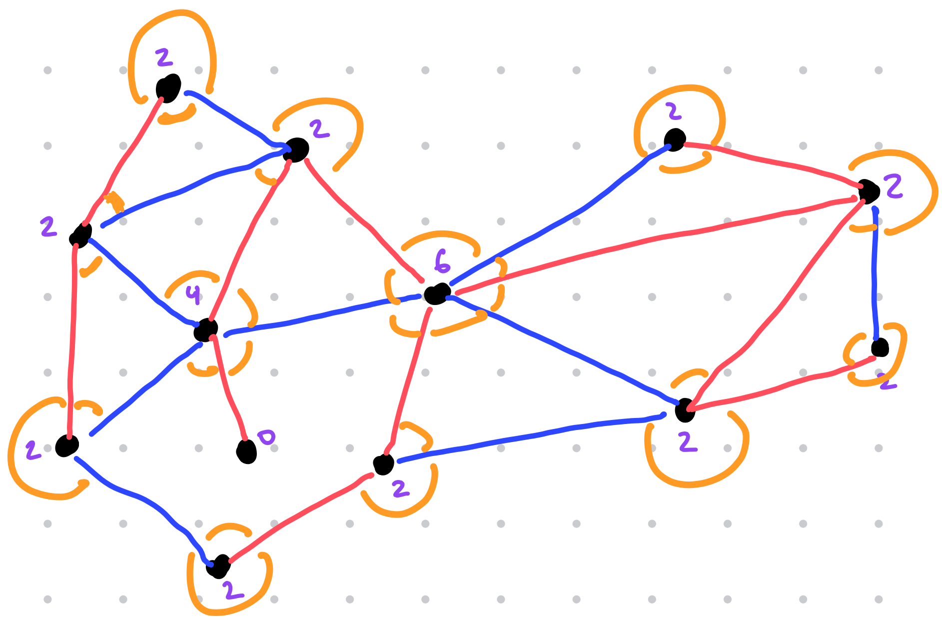
corresponding dihedral angles are the same

Nonconvex Polyhedra: No

Convex Polyhedra: Yes — Cauchy's Rigidity Theorem

Lemma 1:

G



Polyhedral Rigidity

MATH 261 Computational Geometry

1. Complete the proof of the following lemma.

Lemma 1. Let G be a planar graph with edges that are 2-colored. Then there is a vertex v of G with at most two color changes in cyclic order around v.

Proof: Suppose that at every vertex v of G there are more than 2 color changes in cyclic order.

by contradiction

(a) Why must be four or more color changes at each vertex?

The number of cyclic color changes must be even.

So, "more than 2" means 4 or more.

(b) Let c be the total number of face angles incident to edges of 2 colors. We will bound c in two different ways. First, why is $4V \leq c$?

Every vertex has at least 4 color changes.

(c) Second, let f_k be the number of faces in G with exactly k edges. Why is

$$c \leq 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + \cdots ?$$

$$c \leq 2f_3 + 4f_4 + 6f_5 + 8f_6 + 10f_7 + \cdots$$

same

(d) Given your previous answer, why is

$$c \leq 2\underbrace{(3f_3 + 4f_4 + 5f_5 + 6f_6 + \cdots)}_{\text{double count of all edges}} - 4\underbrace{(f_3 + f_4 + f_5 + f_6 + \cdots)}_{\text{total number of faces}}?$$

$$c \leq 2(\quad 2E \quad) - 4(\quad F \quad)$$

(e) Explain how this gives an upper bound on c in terms of E and F.

$$4V \leq c \leq 4E - 4F$$

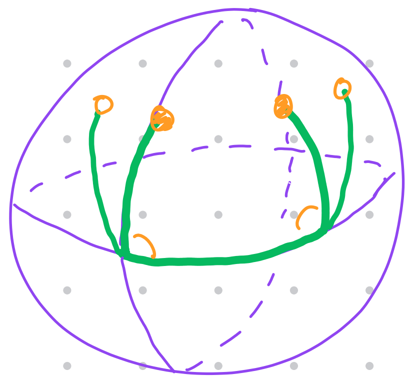
(f) Explain how you now have a contradiction, which proves the lemma.

$$4V \leq 4E - 4F$$

$$V \leq E - F$$

$$V - E + F \leq 0$$ contradicts Euler's formula

SPHERICAL POLYGON: A polygon on the surface of a sphere. Its sides are arcs of great circles



SPHERICAL CONVEX CHAIN:

Take a convex spherical polygon and remove an edge.

2. Draw a picture illustrating the following lemma.

Lemma 2. If a spherical convex chain is opened by increasing some or all of its internal angles, but not beyond π, then the distance between its endpoints is strictly increased.

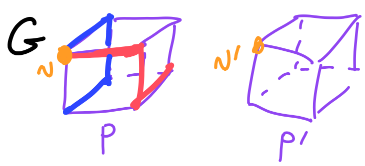
3. Complete the following proof of Cauchy's Rigidity Theorem.

Theorem. If two convex polyhedra are combinatorially equivalent, with corresponding faces congruent and similarly arranged around each vertex, then the dihedral angles at corresponding edges are the same.

Proof: by contradiction Let P and P' be convex polyhedra that are combinatorially equivalent, with corresponding faces congruent and similarly arranged around each vertex, but not congruent.

(a) Color each edge e of P blue or red if the dihedral angle at e is larger or smaller, respectively, than the its corresponding edge of P'. (If the angles are the same, then don't color e.) Let G be the subgraph of the 1-skeleton of P containing the colored edges.

Why is there a vertex v of G with at most two color changes in cyclic order around v?



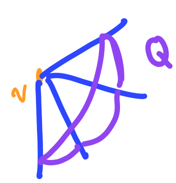
by Lemma 1.

(b) Let S_v be a small sphere centered at v. The faces of P intersected with S_v form a convex spherical polygon Q. Similarly, the faces of P' intersected with a small sphere centered at v' form a spherical polygon Q'. What is the same about Q and Q'? What is different?

Q and Q' have arcs of identical lengths in the same order.

Angles could be different.

(c) If there are no color changes around v, what contradiction can be derived using Lemma 2?



If you increase all angles in Q, some arc must get longer.

(d) If there are exactly two color changes around v, what contradiction can be derived using Lemma 2?