from last time:

5. Suppose that a patient is tested for a disease. Let A be the event that the test is positive, and let D be the event that the patient actually has the disease. Further suppose that:

$P(A \mid D) = 0.99$	(*sensitivity*: probability of a positive test if the patient has the disease)
$P(A' \mid D') = 0.99$	(*specificity*: probability of a negative test if the patient doesn't have the disease)

(a) *Rare disease*: If $P(D) = 0.01$, what is the probability that a patient who tests positive actually has the disease?

$$P(D \mid A) \underset{\text{Bayes' Theorem}}{=} \frac{P(A \mid D)\,P(D)}{P(A)} \underset{\text{Law of Total Probability}}{=} \frac{P(A \mid D)\,P(D)}{P(A \mid D)P(D) + P(A \mid D')P(D')} = \frac{(0.99)(0.01)}{(0.99)(0.01) + (0.01)(0.99)} = \frac{1}{2}$$

Imagine testing 1000 people: 990 without disease, 10 have the disease

- of 10 with disease, with high probability all test positive
- of 990 without disease, about 10 tes positive anyway

$\frac{10}{20}$ positive tests actually have disease

(b) *Common disease*: If $P(D) = 0.1$, what is the probability that a patient who tests positive actually has the disease?

$$P(D \mid A) = \frac{P(A \mid D)P(D)}{P(A \mid D)P(D) + P(A \mid D')P(D')} = \frac{(0.99)(0.1)}{(0.99)(0.1) + (0.01)(0.9)} = 0.917$$

Now: test 1000 people, 900 have the disease:

- 100 with disease $\Rightarrow$ 99 test positive
- 900 without disease $\Rightarrow$ 9 false positives

$\frac{99}{108} = 0.917$

NEW WORKSHEET (Section 1.5)

1. A fair red die and a blue die are rolled. Let A be the event that the red die rolls 2, let B be the event that the sum of the rolls is 5, and let C be the event that the sum of the rolls is 7.

(a) Find $P(A)$ and $P(A \mid B)$. If you know whether B occurs, does it affect your assessment of the probability that A also occurs?

$P(A) = \frac{1}{6}$ $P(A \mid B) = \frac{1}{4}$ A, B are DEPENDENT

Knowledge about B affects our assessment of the prob. of A.

(b) Now find $P(A \mid C)$. If you know whether C occurs, does it affect your assessment of the probability that A also occurs?

$P(A \mid C) = \frac{1}{6}$ A, C are INDEPENDENT

Knowledge about C does not affect assessment of the prob. of A.

DEFINITION: A and B are **INDEPENDENT** if $P(A \mid B) = P(A)$, and are **DEPENDENT** otherwise.

PROPOSITION: A and B are independent if and only if $P(A \cap B) = P(A)P(B)$.

2. A sequence of n independent trials are to be performed. Each trial results in a success with probability p and a failure with probability $1 - p$. What is the probability that…

(a) …all trials result in successes?

Let A_i be the event that trial i results in success.

Then: $P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n) = \underbrace{p \cdot p \cdot \cdots \cdot p}_{n \text{ factors}} = p^n$

(b) …at least one trials results in a success?

Consider the complement: Probability of no successes: $P(A_1' \cap A_2' \cap \cdots \cap A_n') = (1-p)^n$

Thus, the prob. of at least one success: $1 - (1-p)^n$

(c) …exactly k trials result in successes?

There are $\binom{n}{k}$ sequences of n trials with exactly k successes.

Any particular such sequence occurs with probability $p^k (1-p)^{n-k}$

k successes $n-k$ failures

Thus: $P(\text{exactly } k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}$

DEFINITION: Events $A_1, \ldots, A_n$ are **MUTUALLY INDEPENDENT** if for every $k \in \{2, 3, \ldots, n\}$ and every subset $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k}).$$

IN ENGLISH: Events are mutually independent if the probability of any subset of the events is the product of the individual probabilities.

3. Consider an urn containing four balls, numbered 110, 101, 011, and 000. One ball is drawn at random. For $k = 1,2,3$, let A_k be the event that the k^{th} digit is a 1 on the ball that is drawn.

(a) Are the events A_1, A_2, and A_3 pairwise independent? Why or why not?

YES: $P(A_i) = \frac{1}{2}$ for any $i \in \{1,2,3\}$

$P(A_i \cap A_j) = P(A_i)P(A_j)$

$P(A_i \cap A_j) = \frac{1}{4}$ for $i,j \in \{1,2,3\}$

(b) Are the events A_1, A_2, and A_3 mutually independent? Why or why not?

NO: $P(A_1 \cap A_2 \cap A_3) \stackrel{?}{=} P(A_1)P(A_2)P(A_3)$

$0 \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$