POISSON DISTRIBUTION

If $X \sim \text{Poisson}(\mu)$, then: $P(X=x) = p(x;\mu) = e^{-\mu}\frac{\mu^x}{x!}$ for $x \in \{0,1,2,\ldots\}$

and $E(X) = \mu$ and $Var(X) = \mu$.

1. Suppose that the number of phone calls an office receives has a Poisson distribution with a mean of 5 calls per hour.

(a) What is the probability that exactly 7 calls are received between 10:00 and 11:00?

Let $X \sim \text{Poisson}(5)$.

Then $P(X=7) = e^{-5}\frac{5^7}{7!} = 0.104$ R: dpois(7, 5)

(b) What is the probability that more than 7 calls are received between 10:00 and 11:00?

$P(X>7) = 1 - \sum_{x=0}^{7} e^{-5}\frac{5^x}{x!}$

$= \sum_{x=8}^{\infty} e^{-5}\frac{5^x}{x!} \approx 0.133$

R: $1 - \underbrace{\text{ppois}(7,5)}_{P(X \le 7)}$

(c) What is the probability that exactly 10 calls are received between 10:00 and 12:00?

Let X_1 and $X_2 \sim \text{Poisson}(5)$. Then $P(X_1+X_2=10) = \sum_{n=0}^{10} P(X_1=n)\,P(X_2=10-n) = \sum_{n=0}^{10}\left(e^{-5}\frac{5^n}{n!}\right)\left(e^{-5}\frac{5^{10-n}}{(10-n)!}\right)$

or: Let $Y \sim \text{Poisson}(10)$. Then Y is the number of calls in 2 hours.

$P(Y=10) = e^{-10}\frac{10^{10}}{10!} = 0.125$ R: dpois(10, 10)

2. Suppose that a machine produces items, 2% of which are defective. Let X be the number of defective items among 500 randomly-selected items produced by the machine.

(a) What is the distribution of X?

$X \sim \text{Bin}(\underset{n}{500}, \underset{p}{0.02})$

(b) What are the mean and variance of X?

$E(X) = np = 500(0.02) = 10$

$Var(X) = np(1-p) = 500(0.02)(0.98) = 9.8$

(c) What is $P(X=12)$?

$P(X=12) = \binom{500}{12}(0.02)^{12}(0.98)^{488} = 0.0955$

(d) What Poisson distribution approximates the distribution of X?

Poisson with mean 10.

Let $Y \sim \text{Poisson}(10)$.

similar

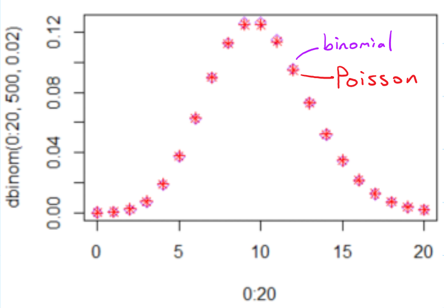
Let $Y \sim \text{Poisson}(10)$.

(e) Use your Poisson distribution to approximate $P(X = 12)$?

$$\underbrace{P(X=12)}_{\text{binomial}} \approx \underbrace{P(Y=12)}_{\text{Poisson}} = e^{-10}\frac{10^{12}}{12!} \approx 0.0948$$

NOTE: In a binomial experiment with large n, $\underbrace{b(x;n,p)}_{\text{binomial}} \approx \underbrace{p(x;np)}_{\text{poisson}}$. $\mu = np$

Approximation is very good if n is big (say, $n \ge 20$) and p is small (say, $np \le 10$).

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> plot(0:20, dbinom(0:20, 500, .02), col="purple", pch=5)
> points(0:20, dpois(0:20, 10), col="red", pch=8)
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3. Suppose that there are N balls in an urn; M of the balls are green, and the rest are not green.

(a) How many ways can n balls be chosen from the urn, if the chosen balls include exactly x green balls?

choose x green balls from M — choose $n-x$ non-green balls from $N-M$

$$\binom{M}{x}\binom{N-M}{n-x}$$

[Selection without replacement]

(b) If n balls are chosen from the urn at random, what is the probability that exactly x green balls will be chosen?

$$P(x \text{ green balls chosen}) = \boxed{\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}}$$

Hypergeometric probability mass formula

total number of ways to choose n balls

(c) Given N, M, and n, what are the bounds on the possible values of x?

$0 \le x$, $\quad x \le M$,

$n - x \le N - M$, $\quad x \le n$