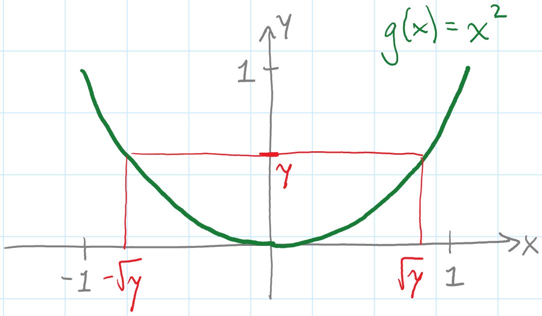
3. Let X have pdf $f_X(x) = \frac{x+1}{2}$ for $-1 \le x \le 1$. Find the density of $Y = X^2$.



CDF method:

$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

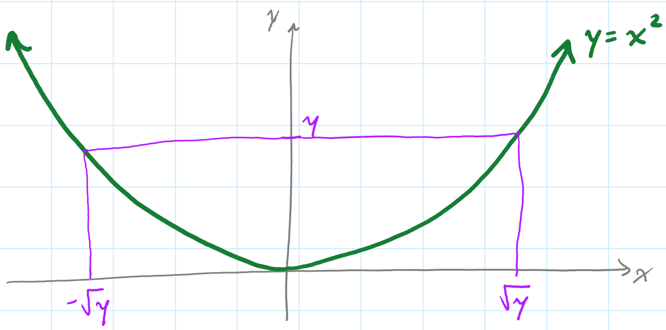
$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{x+1}{2}\,dx$$

$$= \left[\frac{x^2}{4} + \frac{x}{2}\right]_{-\sqrt{y}}^{\sqrt{y}} = \left(\cancel{\frac{y}{4}} + \frac{\sqrt{y}}{2}\right) - \left(\cancel{\frac{y}{4}} - \frac{\sqrt{y}}{2}\right)$$

$$F_Y(y) = \sqrt{y}$$

So: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(\sqrt{y}) = \boxed{\frac{1}{2\sqrt{y}} \text{ for } 0 \le y \le 1}$

1. Let $X \sim N(0,1)$ and $Y = X^2$. What is the distribution of Y?



Now, $X \in \mathbb{R}$ so $Y \in [0, \infty)$.

Find the cdf of Y.

For $y \ge 0$:

$$F_Y(y) = P(Y \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

[FTC: $\frac{d}{dy} \int_{\alpha(y)}^{\beta(y)} f(x)\,dx = f(\beta(y))\beta'(y) - f(\alpha(y))\alpha'(y)$]

$$\text{So } f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}} - \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{-1}{2\sqrt{y}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2\sqrt{y}} \left(e^{-y/2} + e^{-y/2}\right) \quad \text{so} \quad \boxed{f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \text{ for } y \ge 0}$$

This is a Gamma density function with $\alpha = \frac{1}{2}$ and $\beta = 2$.
This is also called the chi-square distribution with 1 degree of freedom.

QUESTION: Given a pdf $f_X(x)$, how can we simulate random variables with this density?

IDEA: Generate Unif[0,1] values and transform them... but what is the transformation function?

Let $U \sim \text{Unif}[0,1)$, so U has pdf $f_U(u) = 1$ for $0 \le u < 1$

Want a function $g(u)$ such that $X = g(U)$ has pdf $\underbrace{f_X(x)}_{\text{Known}}$.

Assume $g(u)$ is strictly incr, and apply the Transformation Theorem:

$$f_X(x) = \underbrace{f_U(h(x))}_{1} \, \Big|\underbrace{h'(x)}_{h'>0}\Big| \quad \text{where } h \text{ is the inverse of } g.$$

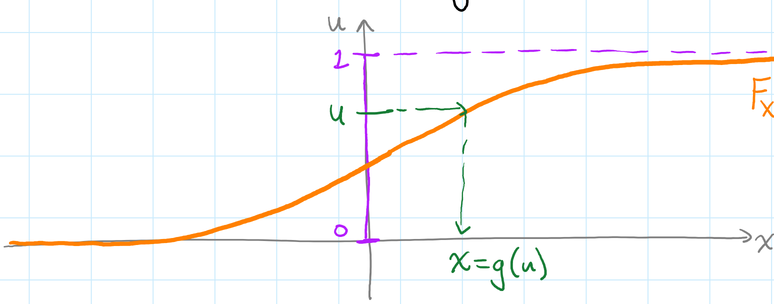
since g incr, so is h

simplify:

$$f_X(x) = h'(x)$$ so $h(x)$ is the antiderivative of $f_X(x)$,

or $h(x) = F_X(x)$.

Thus, $g(x)$ is the inverse of $F_X(x)$, the cdf of X.

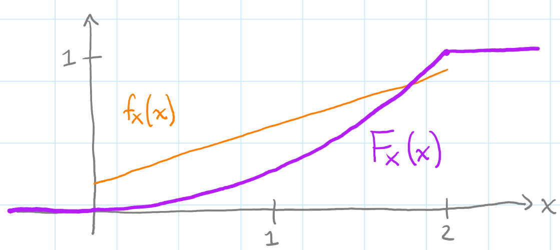


This is the same as the method from chapter 2!

2. Let X have density given by $f_X(x) = \frac{1}{8} + \frac{3}{8}x$ for $0 \le x \le 2$ and 0 otherise. Write a program to simulate values from this distribution.

(a) What is the cdf $F_X(x)$?

$$F_X(x) = \int_0^x \left(\tfrac{1}{8} + \tfrac{3}{8}t\right) dt = \frac{x}{8} + \frac{3x^2}{16}$$



(b) What is the inverse of $F_X(x)$?

$$u = \frac{x}{8} + \frac{3x^2}{16}$$

$$\frac{16}{3}u = x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}$$

$$\frac{16}{3}u = \left(x + \frac{1}{3}\right)^2 - \frac{1}{9}$$

$$\frac{16}{3}u + \frac{1}{9} = \left(x + \frac{1}{3}\right)^2$$

$$\sqrt{\frac{16}{3}u + \frac{1}{9}} = x + \frac{1}{3}$$

$$\boxed{x = \sqrt{\frac{16}{3}u + \frac{1}{9}} - \frac{1}{3}}$$

(c) Write a program to simulate values from this distribution.

```
> uvals <- runif(100000)
> hist(uvals)
> xvals <- sqrt(16*uvals/3 + 1/9) - 1/3
> hist(xvals)
```

3. Let $X \sim \text{Exp}(\lambda)$. Write a program to simulate values of X.

X has pdf $\lambda e^{-\lambda x}$ and cdf $F_X(x) = 1 - e^{-\lambda x}$.

Invert: $u = 1 - e^{-\lambda x}$

$e^{-\lambda x} = 1 - u$

$-\lambda x = \ln(1-u)$

$$\boxed{x = \frac{-1}{\lambda}\ln(1-u)}$$

```
> uvals <- runif(100000)
> xvals <- -log(1-uvals)
> hist(xvals)
```

4. Can you think of distributions that would be impossible to simulate using the inverse CDF method?

- might be impossible to write a formula for the cdf — eg. Normal cdf
- might be impossible to write a formula for the inverse of the cdf