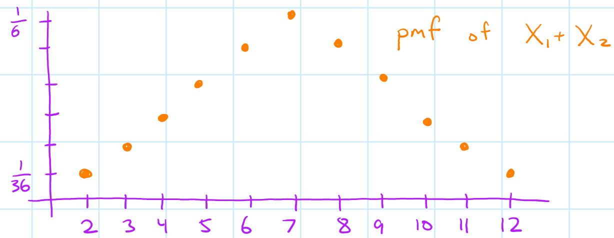
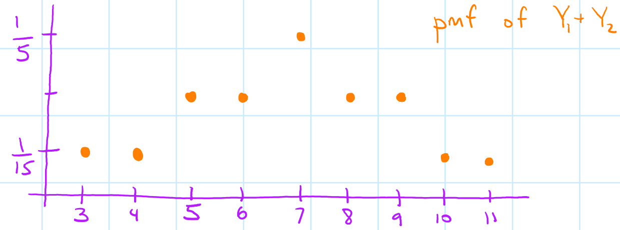
FROM LAST TIME:

X_1 and X_2 are values on two dice (independent)



Y_1 and Y_2 are numbers on two balls selected without replacement (dependent)



DISTRIBUTION OF A SUM

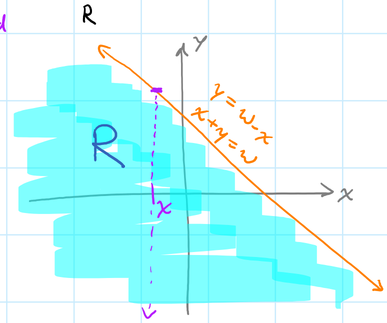
Let X and Y be independent continuous rvs with pdfs $f_X(x)$ and $f_Y(y)$.

What is the density (pdf) of $W=X+Y$?

independence: $f(x,y)=f_X(x)f_Y(y)$

CDF of W: $F_W(w)=P(W\le w)=P(X+Y\le \underset{\text{fixed}}{w})=\iint_R f_X(x)f_Y(y)\,dA$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{w-x} f_X(x)f_Y(y)\,dy\,dx \qquad x+y\le w$$



$$=\int_{-\infty}^{\infty} f_X(x)\left(\int_{-\infty}^{w-x} f_Y(y)\,dy\right)dx$$

$$F_W(w)=\int_{-\infty}^{\infty} f_X(x)\,F_Y(w-x)\,dx$$

Differentiate $F_W(w)$ with respect to w:

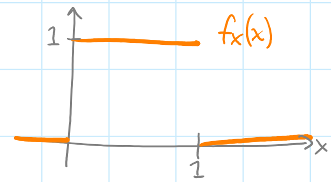
$$f_W(w)=\frac{d}{dw}\int_{-\infty}^{\infty} f_X(x)\,F_Y(w-x)\,dx=\boxed{\int_{-\infty}^{\infty} f_X(x)f_Y(w-x)\,dx}$$

This is called the CONVOLUTION of f_X and f_Y.

[More generally: convolution of f and g is:
$$(f*g)(x)=\int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt$$]

1. Let X and Y be independent uniform variables on $[0,1]$, and let $W = X+Y$. What is the density of W?

NOTE: $0 \le W \le 2$



We have: $f_W(w) = \int_{-\infty}^{\infty} \underbrace{f_X(x)}_{} \underbrace{f_Y(w-x)}_{} \, dx$

$f_X(x) = 1$ if $0 \le x \le 1$, otherwise 0

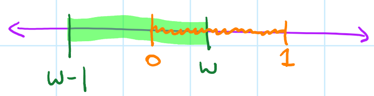
$f_Y(w-x) = 1$ if $0 \le w-x \le 1$, otherwise 0

$-w \le -x \le 1-w$

$w-1 \le x \le w$

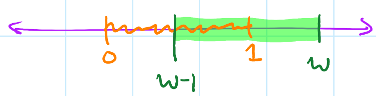
Integrand is 1 iff both hold.

If $0 \le w \le 1$:



$f_W(w) = \int_0^w 1 \, dx = w$

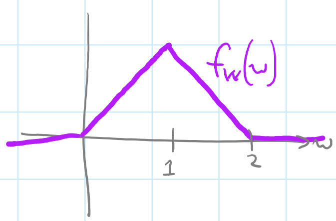
If $1 \le w \le 2$:



$f_W(w) = \int_{w-1}^{1} 1 \, dx = x \Big|_{w-1}^{1} = 1-(w-1) = 2-w$

Density of W:

$$f_W(w) = \begin{cases} w & \text{if } 0 \le w \le 1 \\ 2-w & \text{if } 1 < w \le 2 \\ 0 & \text{otherwise} \end{cases}$$



2. Let $X \sim \text{Exp}(3)$, $Y \sim \text{Exp}(2)$, and $W = X+Y$. What is the density of W?

$X \ge 0 \quad Y \ge 0$, so $W \ge 0$

pdfs: $f_X(x) = 3e^{-3x}$, $\quad f_Y(y) = 2e^{-2y}$

pdf of W: $f_W(w) = \int_{-\infty}^{\infty} \underbrace{f_X(x) f_Y(w-x)}_{\text{only nonzero if}} \, dx = \int_0^w \left(3e^{-3x}\right)\left(2e^{-2(w-x)}\right) dx$

$\underbrace{0 \le x}$ and $w - x \ge 0$

$\underbrace{x \le w}$

$f_W(w) = 6e^{-2w}\left(1 - e^{-w}\right)$ for $w \ge 0$.