MGFs and SUMS of RVs

If X and Y are independent rvs and $W = X + Y$, then

$$M_W(t) = M_X(t)\, M_Y(t).$$

why? $M_W(t) = E(e^{tW}) = E(e^{t(X+Y)}) = E(e^{tX} e^{tY}) = E(e^{tX})\, E(e^{tY}) = M_X(t)\, M_Y(t)$

more generally: if $W = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n + b$,

then $M_W(t) = e^{bt} M_{X_1}(a_1 t)\, M_{X_2}(a_2 t) \cdots M_{X_n}(a_n t)$ } generalizes $M_{aX+b}(t) = e^{bt} M_X(at)$

3. Let $X \sim \text{Bin}(n_1, p)$ and $Y \sim \text{Bin}(n_2, p)$, and X and Y are independent.

(a) What do you think is the distribution of $X + Y$?

$X+Y$ counts successes in a total of n_1+n_2 trials, prob. of success is p for each trial. So $X+Y \sim \text{Bin}(n_1+n_2, p)$

(b) Use mgfs to find the distribution of $X + Y$.

$$M_X(t) = (1-p+pe^t)^{n_1} \quad \text{and} \quad M_Y(t) = (1-p+pe^t)^{n_2}$$

$$M_{X+Y}(t) = M_X(t)\, M_Y(t) = (1-p+pe^t)^{n_1}(1-p+pe^t)^{n_2} = \underbrace{(1-p+pe^t)^{n_1+n_2}}$$

This is the mgf for $\text{Bin}(n_1+n_2, p)$.

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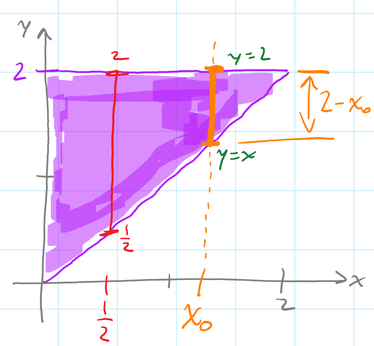
1. Let X and Y have joint density $f(x, y) = \frac{1}{2}$ for $0 \le x \le y \le 2$.



(a) Suppose you know that $X = x_0$. What is then the density of Y?

If $X = x_0$ is fixed, then $Y \sim \text{Unif}[x_0, 2]$.

Then $\underbrace{f_{Y|X}(y|x_0)}_{\text{conditional density of } Y \text{ given } X=x_0} = \dfrac{1}{2-x_0}$ for $x_0 \le y \le 2$.

(b) In part (a), you found the conditional density $f_{Y|X}(y \mid x_0)$. How does this relate to the joint density $f(x, y)$ and the marginal density $f_X(x)$?

Marginal density of X: $f_X(x) = \displaystyle\int_x^2 f(x,y)\, dy = \int_x^2 \tfrac{1}{2}\, dy = \frac{y}{2}\Big|_x^2 = \frac{2}{2} - \frac{x}{2} = \frac{2-x}{2}$

for $0 \le x \le 2$

Observe: $\dfrac{f(x,y)}{f_X(x)} = \dfrac{\frac{1}{2}}{\frac{2-x}{2}} = \dfrac{1}{2-x} = f_{Y|X}(y|x)$

DEFINITION OF CONDITIONAL PDF: $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$

Recall conditional probability of events: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ similar!

(c) If $X = \frac{1}{2}$, then what is the probability that $Y \leq 1$?

$$P(Y \leq 1 \mid X = \tfrac{1}{2}) = \int_{\frac{1}{2}}^{1} f_{Y|X}(y \mid \tfrac{1}{2})\, dy = \int_{\frac{1}{2}}^{1} \frac{1}{2-\frac{1}{2}}\, dy = \int_{\frac{1}{2}}^{1} \frac{2}{3}\, dy = \frac{1}{3}$$

NOTE: $\frac{1}{2} \leq Y \leq 1$

(d) What is the expected value of Y given that $X = x_0$?

$$E(Y \mid X = x_0) = \int_{x_0}^{2} y \cdot f_{Y|X}(y \mid x_0)\, dy = \int_{x_0}^{2} y \cdot \frac{1}{2-x_0}\, dy = \frac{y^2}{2} \cdot \frac{1}{2-x_0} \Big|_{y=x_0}^{y=2}$$

$$= \frac{4}{2} \cdot \frac{1}{2-x_0} - \frac{x_0^2}{2} \cdot \frac{1}{2-x_0} = \frac{4-x_0^2}{2(2-x_0)} = \frac{2+x_0}{2} \quad \text{for } x_0 \leq y \leq 2$$

CONDITIONAL EXPECTATION of Y given $X = x_0$:

$$\mu_{Y|X=x_0} = E(Y \mid X = x_0) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x_0)\, dy$$

(e) What is the variance of Y given that $X = x_0$?

CONDITIONAL VARIANCE: $Var(Y \mid X = x_0) = E(Y^2 \mid X = x_0) - E(Y \mid X = x_0)^2$

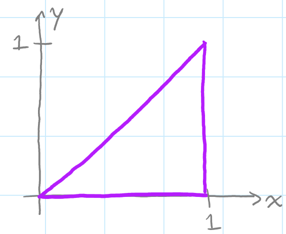
$$E(Y^2 \mid X = x_0) = \int_{x_0}^{2} y^2 f_{Y|X}(y \mid x_0)\, dy = \int_{x_0}^{2} y^2 \frac{1}{2-x_0}\, dy = \frac{y^3}{3} \cdot \frac{1}{2-x_0} \Big|_{x_0}^{2} = \frac{4+2x_0+x_0^2}{3}$$

$$Var(Y \mid X = x_0) = \frac{4+2x_0+x_0^2}{3} - \left(\frac{2+x_0}{2}\right)^2 = \frac{(2-x_0)^2}{12}$$

We didn't do #2 in class, but it's here as another example:

2. The joint pdf of X and Y is $f(x, y) = 3x$, for $0 \le y \le x \le 1$.

(a) What is the conditional distribution of Y given $X = x$?



$$f_X(x) = \int_0^x 3x \, dy = 3xy \Big|_{y=0}^{y=x} = 3x^2 \text{ for } 0 \le x \le 1$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{3x}{3x^2} = \frac{1}{x} \text{ for } 0 \le y \le x$$

(b) What is $E(Y \mid X = x)$?

$$E(Y \mid X = x) = \int_0^x y \cdot \frac{1}{x} \, dy = \frac{1}{x} \cdot \frac{y^2}{2} \Big|_{y=0}^{y=x} = \frac{x}{2}$$

(c) What is $\mathrm{Var}(Y \mid X = x)$?

$$E(Y^2 \mid X = x) = \int_0^x y^2 \cdot \frac{1}{x} \, dy = \frac{1}{x} \cdot \frac{y^3}{3} \Big|_{y=0}^{y=x} = \frac{x^2}{3}$$

$$\mathrm{Var}(Y \mid X = x) = \frac{x^2}{3} - \left(\frac{x}{2}\right)^2 = \frac{x^2}{3} - \frac{x^2}{4} = \frac{x^2}{12}$$

(d) Are X and Y independent?

No, because the possible values of Y depend on X.