From last time:

(WEAK) LAW OF LARGE NUMBERS

If $X_1, X_2, \ldots, X_n$ are iid rvs with mean $\mu < \infty$, and $\overline{X}_n = \frac{X_1 + \cdots + X_n}{n}$, then for any $\varepsilon > 0$, $\lim_{n\to\infty} P(|\overline{X}_n - \mu| \geq \varepsilon) = 0$.

INTERPRETATION: As $n \to \infty$, it's increasingly unlikely that $\overline{X}_n$ will differ from μ by ε.

4. Suppose you flip a fair coin lots of times. What does the Law of Large Numbers say about the numbers of heads and tails you will observe?

Choose $\varepsilon > 0$. There is a high probability that the observed proportion of heads will be within ε of $\frac{1}{2}$, and this probability goes to 1 as number of coin flips incr.

5. Suppose that a certain casino game costs \$1 to play, and the expected winnings per game are \$0.98. What does the Law of Large Numbers say about your winnings if you play the game lots of times?

In the long run, it's almost guaranteed that you will lose $2n$ cents, where n is the number of games you play.

TRANSFORMATIONS OF JOINT DISTRIBUTIONS

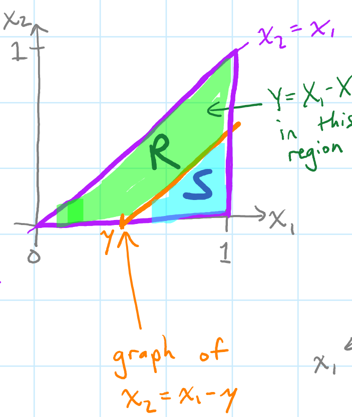
Today: X_1 and X_2 are jointly distributed, and $Y = f(X_1, X_2)$.
What is the distribution of Y?

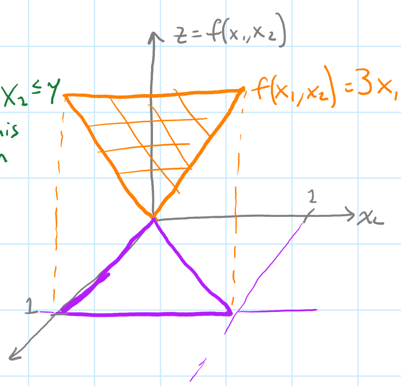
1. Let X_1 and X_2 have joint density $f(x_1, x_2) = 3x_1$, for $0 \leq x_2 \leq x_1 \leq 1$. Let $Y = X_1 - X_2$. Use the following steps to find the density of Y.

(a) Identify the possible values of Y.

$0 \leq Y \leq 1$

Since $Y = X_1 - X_2$ and $0 \leq X_2 \leq X_1 \leq 1$.





(b) Find the region in the x_1x_2-plane where $Y = y$.

Let $y \in [0,1]$. $Y = y$

$X_1 - X_2 = y$ so $X_2 = X_1 - y$

fixed

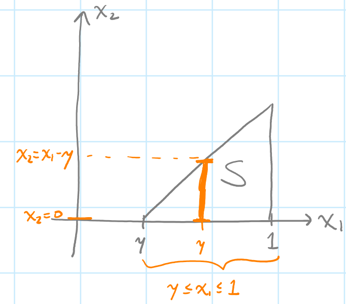
(c) Find the region R in the x_1x_2-plane where $Y \le y$.

R is shaded green in the diagram

(d) Find the cdf $F_Y(y)$ by integrating the joint density of X_1 and X_2 over the region R.

cdf: $F_Y(y) = P(Y \le y) = P(X_1 - X_2 \le y) = \iint_R f(x_1, x_2)\, dA$

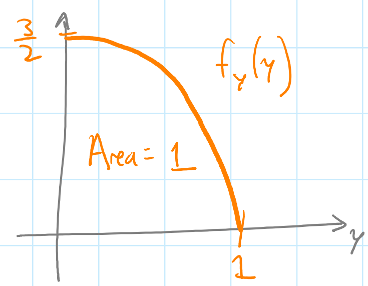
$= 1 - \iint_S f(x_1, x_2)\, dA = 1 - \int_y^1 \int_0^{x_1 - y} 3x_1 \, dx_2\, dx_1$



$F_Y(y) = \frac{3}{2}y - \frac{1}{2}y^3$

(e) Differentiate $F_Y(y)$ to obtain the density $f_Y(y)$.

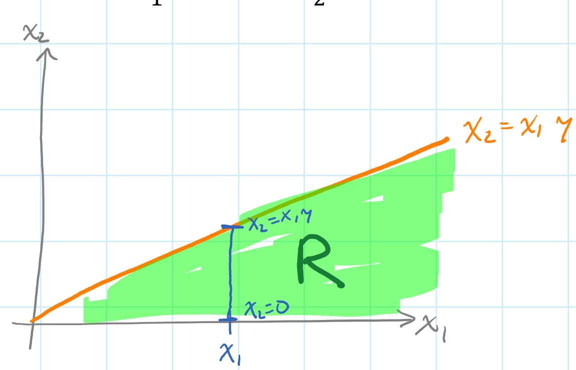
$f_Y(y) = \frac{d}{dy}\left(F_Y(y)\right) = \frac{d}{dy}\left(\frac{3}{2}y - \frac{1}{2}y^3\right) = \frac{3}{2} - \frac{3}{2}y^2$ for $0 \le y \le 1$



2. The joint density of X_1 and X_2 is given by $f(x_1, x_2) = \frac{1}{8}x_1 e^{-(x_1+x_2)/2}$ for $x_1 > 0$ and $x_2 > 0$. Find the density function for the ratio $Y = \frac{X_2}{X_1}$.

(a) Identify the possible values of Y.

$Y = \frac{X_2}{X_1}$ $Y > 0$



(b) Find the region in the x_1x_2-plane where $Y = y$.

Fix $y > 0$. Then $Y = \frac{X_2}{X_1} = y$

when $X_2 = X_1 y$.   line through the origin with slope y

(c) Find the region R in the x_1x_2-plane where $Y \le y$.

$Y \le y$ when $X_2 \le X y$

(d) Find the cdf $F_Y(y)$ by integrating the joint density of X_1 and X_2 over the region R.

$$F_Y(y) = P(Y \le y) = P\left(\frac{X_2}{X_1} \le y\right) = \iint_R f(x_1, x_2)\, dA$$

$$= \int_0^\infty \int_0^{x_1 y} \frac{1}{8} x_1 e^{-(x_1+x_2)/2}\, dx_2\, dx_1 = \frac{y^2 + 2y}{(1+y)^2} \quad \text{for } y > 0$$

(e) Differentiate $F_Y(y)$ to obtain the density $f_Y(y)$.

$$f_Y(y) = \frac{d}{dy}\left[\frac{y^2+2y}{(1+y)^2}\right] = \frac{2}{(1+y)^3} \quad \text{for } y > 0$$

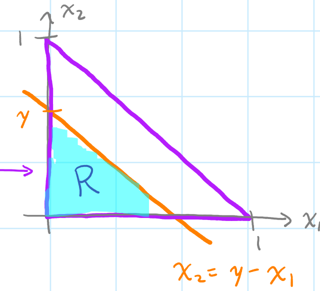
We didn't do the next problem in class, but it's here as another example:

3. Let X_1 and X_2 be uniformly distributed over the region of the x_1x_2-plane defined by $0 \le x_1$, $0 \le x_2$, and $x_1 + x_2 \le 1$. Let $Y = X_1 + X_2$. Find the density of Y.

$0 \le Y \le 1$

For $y \in [0, 1]$: $Y = X_1 + X_2 = y \Rightarrow X_2 = y - X_1$

$Y \le y \Rightarrow X_2 \le y - X_1$ region R



Then, $F_Y(y) = P(X_2 \le y - X_1) = \iint_R f(x_1, x_2)\, dA = \iint_R 2\, dA = 2 \cdot \text{Area}(R) = y^2$

Thus: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(y^2) = 2y$ for $0 \le y \le 1$.