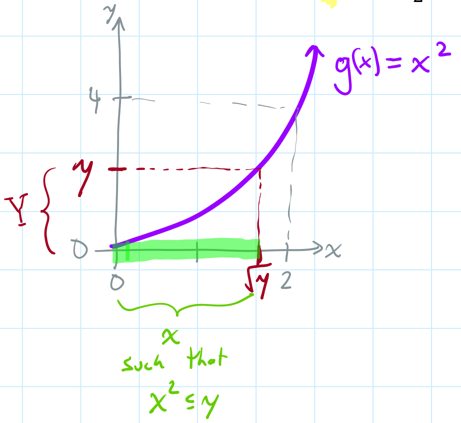
TRANSFORMATIONS OF RANDOM VARIABLES

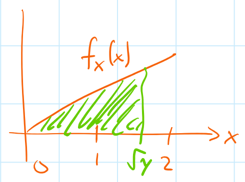
1. Let X have density $f_X(x) = \frac{x}{2}$ for $0 \le x \le 2$, and let $Y = X^2$. What is the density of Y?



Note: Y takes values $0 \le y \le 4$

First, find the cdf of Y:

density of X
$f_X(x) = \frac{x}{2}$ for $0 \le x \le 2$



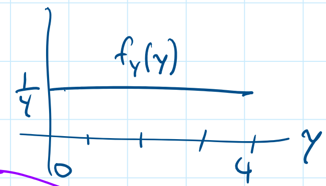
$$F_Y(y) = P(Y \le y) = P(X \le \sqrt{y})$$

$$= \int_0^{\sqrt{y}} \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^{\sqrt{y}} = \frac{(\sqrt{y})^2}{4} - \frac{0}{4}$$

$$F_Y(y) = \frac{y}{4} \quad \text{for} \quad 0 \le y \le 4$$

Differentiate F_Y to obtain the pdf of Y:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(\frac{y}{4}\right) = \frac{1}{4} \quad \text{for} \quad 0 \le y \le 4$$



Transformation Theorem: If X has pdf $f_X(x)$ and $Y = g(X)$ where g is strictly monotonic, then:
(increasing or decreasing)

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$

where h is the inverse function of g.

$$h(y) = g^{-1}(y).$$

In problem 1: $g(x) = x^2$, so inverse function is $h(y) = \sqrt{y}$.

$f_X(x) = \frac{x}{2}$

$h'(y) = \frac{1}{2} y^{-1/2} = \frac{1}{2\sqrt{y}}$

inverse functions
$g(h(y)) = y$
$h(g(x)) = x$

Theorem says:
$$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$
$$= f_X(\sqrt{y}) \cdot \left|\frac{1}{2\sqrt{y}}\right|$$
$$= \frac{\cancel{\sqrt{y}}}{2} \cdot \frac{1}{2\cancel{\sqrt{y}}}$$

$$\boxed{f_Y(y) = \frac{1}{4} \quad \text{for} \quad 0 \le y \le 4}$$