1. A sequence of n independent trials are to be performed. Each trial results in a success with probability p and a failure with probability $1-p$. What is the probability that…

(a) …all trials result in successes?

Let A_i be the event that trial i results in success.

By independence, $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n) = \underbrace{p \cdot p \cdots p}_{n \text{ factors}} = p^n$

(b) …at least one trials results in a success?

Probability of no successes: $P(A_1' \cap A_2' \cap \cdots \cap A_n') = (1-p)^n$

Probability of at least one success: $1 - P(A_1' \cap \cdots \cap A_n') = 1-(1-p)^n$

(c) …exactly k trials result in successes?

We will see this again in Chapter 2.

Any particular sequence of k successes and $n-k$ failures occurs with probability $p^k(1-p)^{n-k}$. There are $\binom{n}{k}$ such sequences.

Thus, $P(\text{exactly } k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}$.

This factor is important!

2. Consider an urn containing four balls, numbered 110, 101, 011, and 000. One ball is drawn at random. For $k \in \{1,2,3\}$, let A_k be the event that the k^{th} digit is a 1 on the ball that is drawn.

(a) Are the events A_1, A_2, and A_3 pairwise independent? Why or why not?

Yes: $P(A_i) = \frac{1}{2}$ for any $i \in \{1,2,3\}$

$P(A_i \cap A_j) = \frac{1}{4}$ for any $i, j \in \{1,2,3\}$

(b) Are the events A_1, A_2, and A_3 mutually independent? Why or why not?

No: $P(A_1 \cap A_2 \cap A_3) = 0 \neq P(A_1)P(A_2)P(A_3)$

3. If A and B are independent events with positive probability, show that they cannot be mutually exclusive.

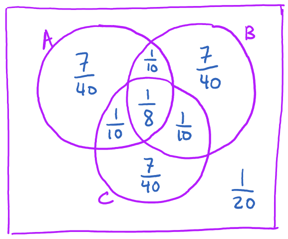
Given: $P(A) > 0$, $P(B) > 0$, and $P(A \cap B) = P(A)P(B)$.

Thus $P(A \cap B) > 0$, so the events <u>can</u> occur simultaneously, meaning they are not mutually exclusive.

4. Create an example of three events A, B, C such that $P(A \cap B \cap C) = P(A)P(B)P(C)$ but the events are not mutually independent.

This is tricky! Trial-and-error is a fine approach.

Here is one example:



$P(A) = P(B) = P(C) = \frac{1}{2}$

$P(A \cap B \cap C) = \frac{1}{8} = P(A)P(B)P(C)$

However:

$P(A \cap B) = \frac{9}{40} \neq P(A)P(B)$

(and similarly for $A \cap C$ and $B \cap C$)

A, B, and C are not pairwise independent, and thus not mutually independent.

BONUS: Suppose that events A and B are independent events, and event C is such that $P(C) > 0$. Is the event A given C independent of the event B and C?

No. Consider the following counterexample.

Two fair coins are flipped.

A is the event that the first coin lands heads.

B is the event that the second coin lands heads.

C is the event that at least one coin lands heads.

The sample space contains four outcomes with equal probability:

$$S = \{HH, HT, TH, TT\}$$

and the first 3 of these outcomes are in event C.

So $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, and $P(A \cap B) = \frac{1}{4} = P(A)P(B)$, which means events A and B are independent.

Now $P(A|C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{\frac{1}{2}}{\frac{3}{4}} = \dfrac{2}{3}$ and similarly $P(B|C) = \frac{2}{3}$.

Also, $P(A \cap B | C) = \dfrac{P(A \cap B \cap C)}{P(C)} = \dfrac{\frac{1}{4}}{\frac{3}{4}} = \dfrac{1}{3}$.

This means that $P(A \cap B | C) = \frac{1}{3} \neq \frac{2}{3} \cdot \frac{2}{3} = P(A|C)P(B|C)$.