LAST TIME: If $X \sim Geo(p)$ and k is a positive integer,

$$P(X > k) = (1-p)^k \quad \leftarrow \text{Geometric tail probability}$$

If $0 < s < t$ (s,t integers), then

$$P(X>t \mid X>s) = \frac{P(X>t \cap X>s)}{P(X>s)} = \frac{P(X>t)}{P(X>s)} = \frac{(1-p)^t}{(1-p)^s}$$

$$= (1-p)^{t-s} = P(X>t-s)$$

Thus: $\underbrace{P(X>t \mid X>s) = P(X>t-s)}$

Memoryless Property of a geometric random variable.

In words: The waiting time until the next success does not depend on the number of failues you have already seen.

MOMENT-GENERATING FUNCTIONS (mgf)

$E(X), E(X^2), E(X^3), \ldots$

moments of a random variable

mgf of random variable X is

def: $M_X(t) = E(e^{tX}) = \sum_x e^{tx} P(X=x)$

$$M_X(t) = 1 + E(X)t + E(X^2)\frac{t^2}{2} + E(X^3)\frac{t^3}{3!} + \cdots$$

To find $E(X^r)$, differentiate $M_X(t)$ r times and evaluate at $t=0$.

EXAMPLE: $X \sim \text{Poisson}(\mu)$

Then: $M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} \underbrace{e^{-\mu}\frac{\mu^k}{k!}}_{P(X=k)} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(\mu e^t)^k}{k!}$

use Taylor Series for e^x

$= e^{-\mu} \underbrace{e^{\mu e^t}}$

$= e^{-\mu + \mu e^t}$

$$\boxed{M_X(t) = e^{\mu(e^t - 1)}}$$ mgf for a Poisson(μ) distribution

Observe: $M_X'(t) = \underbrace{e^{\mu(e^t-1)}} \cdot \underbrace{\mu e^t}$ so $M_X'(0) = e^{\mu(\overset{1}{e^0}-1)} \cdot \mu e^0 = e^0 \cdot \mu e^0 = \mu$

$E(X) = \mu$

$M_X''(t) = \left(e^{\mu(e^t-1)} \mu e^t\right) \cdot \mu e^t + e^{\mu(e^t-1)}\left(\mu e^t\right)$

so $M_X''(0) = \underset{1}{e^{\mu(\overset{1}{e^0}-1)}} \underset{\mu \, 1}{\mu e^0} \cdot \underset{\mu \, 1}{\mu e^0} + \underset{1}{e^{\mu(e^0-1)}} \cdot \underset{\mu \, 1}{\mu e^0}$

$= \mu^2 + \mu$

$E(X^2) = \mu^2 + \mu$