1. Let $X \sim \text{Geometric}(p)$. Compute the mgf $M_X(t)$. Try to write the mgf without a summation.

   (a) What is the mgf $M_X(t)$? Write your answer as a sum.

   $$M_X(t) = E\left(e^{tX}\right) = \sum_{k=1}^{\infty} e^{tk} \underbrace{(1-p)^{k-1}p}_{P(X=k)}$$

   (b) Your summation in part (a) should be a geometric series. Use the geometric sum formula to write the mgf $M_X(t)$ *without* a sum.

   $$\boxed{\text{Geometric series:} \quad \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}$$

   $$M_X(t) = \sum_{k=1}^{\infty} pe^{tk}(1-p)^{k-1} = pe^t + pe^{2t}(1-p) + pe^{3t}(1-p) + pe^{4t}(1-p) + \cdots$$

   first term
   $a = pe^t$

   Common ratio: $r = e^t(1-p)$

   Thus, $\quad M_X(t) = \dfrac{pe^t}{1 - e^t(1-p)}$

   Alternative approach:

   $$M_X(t) = \frac{p}{1-p} \sum_{k=1}^{\infty} e^{tk}(1-p)^k = \frac{p}{1-p} \underbrace{\sum_{k=1}^{\infty}\left(e^t(1-p)\right)^k}_{\text{geometric series: } a=r=e^t(1-p)}$$

   $$= \frac{p}{1-p} \cdot \frac{e^t(1-p)}{1 - e^t(1-p)} = \frac{pe^t}{1 - e^t(1-p)}$$

2. Suppose random variable X has probability mass function $P(X = x) = \frac{27}{40}\left(\frac{1}{3}\right)^x$, for integers $0 \le x \le 3$.

   (a) Verify that this is a valid probability mass function.

   $$\boxed{a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r}}$$

   • nonnegative probabilities

   • Sum is: $\quad \dfrac{27}{40}\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right) = \dfrac{27}{40}\left(\dfrac{1-\left(\frac{1}{3}\right)^4}{1-\frac{1}{3}}\right) = \dfrac{27}{40}\left(\dfrac{\frac{80}{81}}{\frac{2}{3}}\right) = \dfrac{27 \cdot 40}{40 \cdot 27} = 1$

   4 terms of a geometric series with $r = \frac{1}{3}$

(b) Find the moment generating function $M_X(t)$. Try to write the mgf without a summation.

$$M_X(t) = E\left(e^{tX}\right) = e^0 \cdot \frac{27}{40} + e^t \cdot \frac{9}{40} + e^{2t} \cdot \frac{3}{40} + e^{3t} \cdot \frac{1}{40} = \frac{27}{40} \cdot \frac{1 - \left(\frac{e^t}{3}\right)^4}{1 - \left(\frac{e^t}{3}\right)} = \frac{81 - e^{4t}}{40(3 - e^t)}$$

common   ratio: $r = \frac{e^t}{3}$

(c) Compute $M_X'(0)$. Does your answer agree with the expected value of X computed directly from the pmf?

$$M_X(t) = \frac{81 - e^{4t}}{40(3 - e^t)}$$

$$M_X'(t) = \frac{40(3-e^t)(-4e^{4t}) - (81 - e^{4t})(40)(-e^t)}{40^2(3-e^t)^2}$$ Use technology!

$$M_X'(0) = \frac{40(2)(-4) - (80)(40)(-1)}{40^2(2)^2} = \frac{72}{160} = \frac{9}{20}$$

$$E(X) = 0 + 1 \cdot \frac{9}{40} + 2 \cdot \frac{3}{40} + 3 \cdot \frac{1}{40} = \frac{9 + 6 + 3}{40} = \frac{9}{20} \leftarrow \text{Same!}$$

This problem is in the next class video.

3. Let X be a discrete random variable with mgf $M_X(t)$, and let $Y = aX + b$. Write the definition of $M_Y(t)$. Replace Y with $aX + b$ and simplify the expected value to show that $M_Y(t) = e^{tb}M_X(at)$.

$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t(aX+b)}\right) = E\left(e^{atX + bt}\right) = e^{bt}E\left(e^{atX}\right) = e^{bt}M_X(at)$$

If $Y = aX + b$, then $M_Y(t) = e^{bt}M_X(at)$.