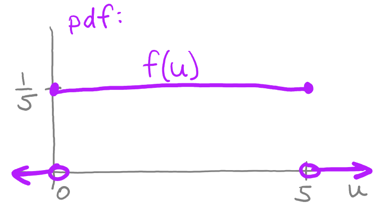

1. Let $U \sim \text{Unif}[0,5]$. — so pdf is $f(u) = \frac{1}{5}$ on $[0,5]$

(a) What are the mean and variance of U?

$U \sim \text{Unif}[A,B]$

$$E(U) = \int_0^5 u \cdot \frac{1}{5}\, du = \frac{1}{10} u^2 \Big|_0^5 = \frac{25}{10} = 2.5 = \frac{B+A}{2}$$

$$E(U^2) = \int_0^5 u^2 \cdot \frac{1}{5}\, du = \frac{1}{15} u^3 \Big|_0^5 = \frac{125}{15} = \frac{25}{3}, \quad \text{so} \quad Var(U) = \frac{25}{3} - \left(\frac{5}{2}\right)^2 = \frac{25}{12} = \frac{(B-A)^2}{12}$$

(b) Let $V = 3U + 2$. What are the mean and variance of V?

RECALL: $Var(aX + b) = a^2\, Var(X)$

$$E(V) = E(3U+2) = 3E(V) + 2 = 3(2.5) + 2 = 9.5$$

$$Var(V) = Var(3U+2) = 3^2\, Var(U) = 9\left(\frac{25}{12}\right) = \frac{75}{4}$$

(c) What do you think is the distribution of V? *Why?*

If U is uniformly distributed on $[0,5]$, and we rescale this interval linearly to $[2,17]$, then it seems that $V = 3U+2$ should be uniformly distributed on $[2,17]$.

2. (a) Let $X \sim \text{Unif}[A,B]$. Use the mgf definition to show that the mgf of X is $M_X(t) = \begin{cases} \frac{e^{Bt} - e^{At}}{(B-A)t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$.

$$M_X(t) = E(e^{tX}) = \int_A^B e^{tx} \frac{1}{B-A}\, dx = \frac{1}{B-A} \cdot \frac{1}{t} e^{tx} \Big|_A^B = \frac{e^{Bt} - e^{At}}{B-A}$$

only if $t \neq 0$

If $t = 0$, $\int_A^B e^0 \frac{1}{B-A}\, dx = \frac{B-A}{B-A} = 1.$

Thus: $M_X(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{e^{Bt} - e^{At}}{(B-A)t} & \text{if } t \neq 0 \end{cases}$ and

(b) Write down the mgfs of U and V from problem 1. How can these mgfs verify your answer for 1(c)?

The mgf of U from problem 1 is: $M_U(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{e^{5t} - 1}{5t} & \text{if } t \neq 0 \end{cases}$

Since $V = 3U + 2$, the mgf for V is:

$$M_V(t) = M_{3U+2}(t) = e^{2t} M_U(3t) = e^{2t} \frac{e^{5(3t)} - 1}{5(3t)} = \frac{e^{17t} - e^{2t}}{15t} \quad \text{if } t \neq 0$$

$$M_V(0) = e^{2t} M_U(0) = 1$$

Note that $M_V(t)$ is the mgf for Unif $[2, 17]$.

Thus, $V \sim$ Unif $[2, 17]$.

3. A stick of length 1 is split at a point U that is uniformly distributed on (0,1).

(a) What is the expected length of the leftmost piece?

The leftmost piece is from 0 to U, so it has length U, and its expected length is $E(U) = \frac{1}{2}$.

(b) What is the expected length of the longest piece?

The two lengths are U and $1-U$, so the length of the longest piece is $\max(U, 1-U)$.

$$E(\max(U, 1-U)) = \int_0^1 \max(u, 1-u) \cdot \underset{\substack{\uparrow \\ \text{pdf of } U}}{1} \, du = \int_0^{\frac{1}{2}} (1-u) \, du + \int_{\frac{1}{2}}^1 u \, du = \left[u - \frac{u^2}{2}\right]_0^{\frac{1}{2}} + \left[\frac{u^2}{2}\right]_{\frac{1}{2}}^1$$

NOTE: $\max(u, 1-u) = \begin{cases} 1-u & \text{if } 0 \le u \le \frac{1}{2}, \\ u & \text{if } \frac{1}{2} < u \le 1. \end{cases}$

$$= \left(\frac{1}{2} - \frac{1}{8}\right) + \left(\frac{1}{2} - \frac{1}{8}\right) = \frac{3}{4}$$

We could also approximate this result via simulation:

```
In[18]:= longest[] := Module[{u},
           u = RandomVariate[UniformDistribution[{0, 1}]];
           Return[Max[u, 1 - u]]
          ]

In[21]:= Mean[Table[longest[], 10000]]

Out[21]= 0.747895
```

(c) What is the expected length of the piece that contains the point p, $0 \le p \le 1$?

Let $L_p(U) = \begin{cases} 1-U & \text{if } U < p, \\ U & \text{if } U > p. \end{cases}$

Then $E(L_p(U)) = \int_0^1 L_p(u)\cdot 1\,du = \int_0^p (1-u)\,du + \int_p^1 u\,du = \left[u - \frac{u^2}{2}\right]_0^p + \left[\frac{u^2}{2}\right]_p^1$

$$= \left(p - \frac{p^2}{2}\right) + \left(\frac{1}{2} - \frac{p^2}{2}\right) = p - p^2 + \frac{1}{2}$$

Observe that $E(L_p(U))$ is maximized when $p = \frac{1}{2}$.

BONUS: Let X be a random variable that takes on values between 0 and c.

(a) Explain why $E(X^2) \le cE(X)$.

Since $x^2 \le cx$ for $x \in [0,c]$, $E(X^2) = \int_0^c x^2 f(x)\,dx \le \int_0^c cx f(x)\,dx = cE(X)$

(b) Use part (a) to show that $\mathrm{Var}(X) \le c^2[\alpha(1-\alpha)]$, where $\alpha = \frac{E(X)}{c}$.

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 \le cE(X) - E(X)^2 = E(X)(c - E(X))$$
$$= c^2\left[\frac{E(X)}{c}\cdot\frac{c - E(X)}{c}\right] = c^2[\alpha(1-\alpha)]$$

(c) Establish an upper bound on $\alpha(1-\alpha)$ and conclude that $\mathrm{Var}(X) \le \frac{c^2}{4}$.

Note that $\alpha(1-\alpha)$ takes a maximum value of $\frac{1}{4}$ when $\alpha = \frac{1}{2}$.

Therefore, $\mathrm{Var}(X) \le c^2[\alpha(1-\alpha)] \le \frac{c^2}{4}$.