$$\begin{tikzcd}[column sep=huge] (X_1, X_2) \arrow[r, bend left, "{\text{Transformation}}", "{u_1,\, u_2}"'] & (Y_1, Y_2) \arrow[l, bend left, "{v_1,\, v_2}", "{\text{inverse transformation}}"'] \end{tikzcd}$$

BIVARIATE TRANSFORMATION THEOREM

Let X_1 and X_2 have joint density $f(x_1, x_2)$.

Let $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$,

with inverse transformation $X_1 = v_1(Y_1, Y_2)$ and $X_2 = v_2(Y_1, Y_2)$.

Let M be the Jacobian matrix:

$$M = \begin{bmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{bmatrix}$$

Then the joint density of Y_1 and Y_2 is given by

$$g(y_1, y_2) = f\Big(\underbrace{v_1(y_1, y_2)}_{x_1},\ \underbrace{v_2(y_1, y_2)}_{x_2}\Big) \cdot \Big|\det(M)\Big|.$$

determinant

1-var. transformation theorem

$$f_Y(y) = f_X\big(h(y)\big) \cdot |h'(y)|$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$