1. Let X_1 and X_2 be iid Exp$\left(\frac{1}{10}\right)$. $\longrightarrow$ $f(x) = \frac{1}{10} e^{-x/10}$ for $x > 0$, $F(x) = 1 - e^{-x/10}$ for $x > 0$

(a) What is the pdf of $Y_1 = \min(X_1, X_2)$?

$$g_1(y) = n[1-F(y)]^{n-1} f(y) = 2\left[1-(1-e^{-y/10})\right]^1 \left(\tfrac{1}{10} e^{-y/10}\right) = 2\left[e^{-y/10}\right]\left(\tfrac{1}{10} e^{-y/10}\right) = \underbrace{\tfrac{1}{5} e^{-y/5}}_{Y_1 \sim \text{Exp}(\lambda = \frac{1}{5})} \text{ for } y > 0$$

(b) What is the expected value of Y_1?

$E(Y_1) = 5$

NOTE: $E(X_1) + E(X_2) = 20 = E(Y_1) + E(Y_2)$

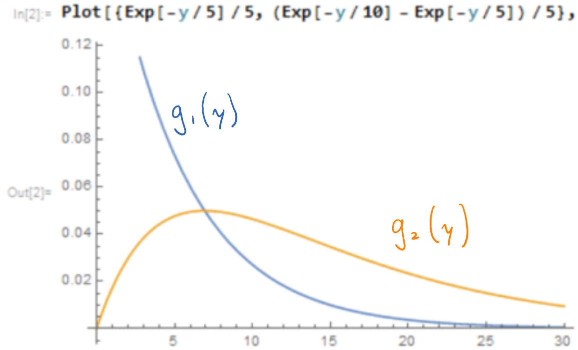
(c) What is the pdf of $Y_2 = \max(X_1, X_2)$? What is $E(Y_2)$?

$$g_2(y) = n[F(y)]^{n-1} f(y) = 2\left[1-e^{-y/10}\right]^1 \left(\tfrac{1}{10} e^{-y/10}\right) = \tfrac{2}{10}\left(e^{-y/10} - e^{-y/5}\right) = \tfrac{1}{5}\left(e^{-y/10} - e^{-y/5}\right) \text{ for } y > 0$$

$$E(Y_2) = \int_0^\infty y \cdot \tfrac{1}{5}\left(e^{-y/10} - e^{-y/5}\right) dy = 15$$

pdfs of Y_1 and Y_2:

```
In[2]:= Plot[{Exp[-y/5]/5, (Exp[-y/10] - Exp[-y/5])/5}, {y, 0, 30}]
```



2. Let X_1, X_2, X_3 be iid Exp$\left(\frac{1}{10}\right)$. What is the expected value of the sample median?

Sample median is Y_2 $\quad (n=3,\ i=2)$

$$g_2(y) = \frac{3!}{1!\,1!}\left[1-e^{-y/10}\right]^1 \left[e^{-y/10}\right]^1 \left(\tfrac{1}{10} e^{-y/10}\right) = \tfrac{6}{10}\left[1-e^{-y/10}\right] e^{-y/5} = \tfrac{3}{5}\left(e^{-y/5} - e^{-3y/10}\right) \text{ for } y > 0$$

$$E(Y_2) = \int_0^\infty y \cdot \tfrac{3}{5}\left(e^{-y/5} - e^{-3y/10}\right) dy = \frac{25}{3}$$

3. Let X_1, X_2, X_3 be iid Unif[0,1]. What is the probability that the sample median is between $\frac{1}{4}$ and $\frac{3}{4}$?

$n = 3$, Unif$[0,1]$ has pdf $f(x) = 1$, cdf $F(x) = x$, for $0 \le x \le 1$

density of the sample median:

$$g_2(y) = \frac{3!}{(2-1)!(3-2)!} y(1-y) = 6y(1-y) \quad \text{for } 0 \le y \le 1$$

Thus, $P\left(\frac{1}{4} < Y_2 < \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} (6y - 6y^2)\,dy = \left[3y^2 - 2y^3\right]_{\frac{1}{4}}^{\frac{3}{4}} = \frac{11}{16} = 0.6875$

4. Let n be a positive odd integer and let $X_1, X_2, \dots, X_n$ be iid Unif[0,1]. What is the smallest n such that the sample median is between 0.4 and 0.6 with probability greater than $\frac{1}{2}$?

median is Y_i with $i = \frac{n+1}{2}$

density: $g_i(y) = \frac{n!}{(i-1)!(n-i)!} y^{i-1}(1-y)^{n-i} = \frac{n!}{\left(\frac{n-1}{2}\right)!^2} y^{\frac{n-1}{2}} (1-y)^{\frac{n-1}{2}}$

Use Mathematica to compute $\int_{0.4}^{0.6} g_i(y)\,dy$ for various n.

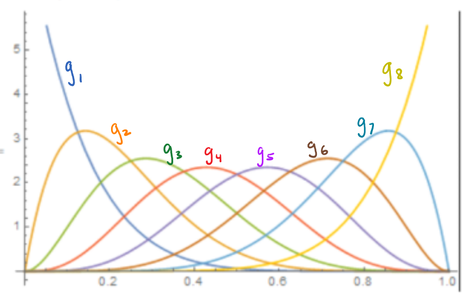
The smallest odd n such that $\int_{0.4}^{0.6} g_i(y)\,dy > \frac{1}{2}$ is $n = 11$.

5. Let $X_1, \dots, X_8$ be iid Unif[0,1].

(a) Make a plot of the pdfs of all eight order statistics.

```
In[16]:= n = 8;
         pdfs = Table[n! / ((i - 1)! (n - i)!) y^(i - 1) (1 - y)^(n - i), {i, 1, n}]
Out[17]= {8 (1 - y)^7, 56 (1 - y)^6 y, 168 (1 - y)^5 y^2, 280 (1 - y)^4 y^3, 280 (1 - y)^3 y^4, 168 (1 - y)^2 y^5, 56 (1 - y) y^6, 8 y^7}

In[18]:= Plot[pdfs, {y, 0, 1}]
Out[18]=
```



(b) What are the expected values of all eight order statistics?

$$E(Y_i) = \frac{i}{9}$$