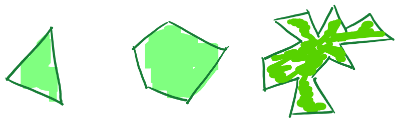
POLYGONS AND TRIANGULATIONS

Def: A polygon is the closed region of the plane bounded by a finite collection of line segments forming a closed curve that does not intersect itself.

Polygons:



Not Polygons:

Let P be a polygon.

DIAGONAL: line segment connecting two vertices of P, lying in the interior of P, not touching boundary ∂P except at endpoints.

TRIANGULATION: A decomposition of P into triangles by a maximal set of noncrossing diagonals

can't add any more

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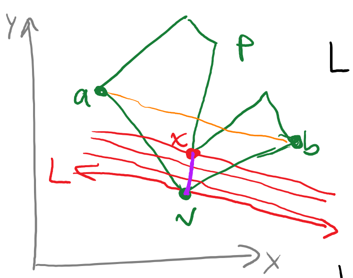
little theorem, used for proving something else

LEMMA: Every polygon with more than 3 vertices has a diagonal.

proof: Let v be the lowest vertex of polygon P (rightmost, if there are multiple lowest vertices.



line sweep argument

Let a and b be neighboring vertices of v.

If segment $\overline{ab}$ is a diagonal, the we are done.

Otherwise, the triangle abv contains some vertex of P.

Let L be the line parallel to $\overline{ab}$ through v.

Sweep L upward toward $\overline{ab}$, and let x be the first vertex that L meets along this sweep. (choose either, if multiple).

Thus, segment $\overline{xv}$ is a diagonal of P.

Sample Proof by Induction

Prove that $\boxed{1+2+3+\cdots+n = \frac{n(n+1)}{2}}$ for all positive integers n.

Call this $S(n)$.

Want to prove this is true for $n = 1, 2, 3, 4, \ldots$

Base case: $S(1)$ is $1 = \frac{1(1+1)}{2}$, which is clearly true.

Induction: Let $k \geq 1$ be an integer, and assume $S(k)$ is true.

[we want to show that $S(k+1)$ is true.]

That is, $1+2+3+\cdots k = \frac{k(k+1)}{2}$. $\leftarrow S(k)$

Add $k+1$ to both sides:

$$1+2+3+\cdots+k+(k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

want: $1+2+3+\cdots+k+(k+1) = \frac{(k+1)(k+1+1)}{2}$ $\leftarrow S(k+1)$

Thus, by mathematical induction, the statement holds for all positive integers n.

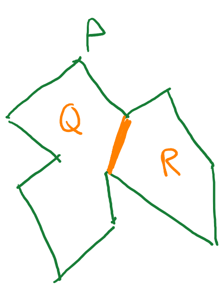
THEOREM: Every polygon has a triangulation.

proof: by induction on the number of vertices n of polygon P

base case: If $n=3$, then P is a triangle, which is its own triangulation.

induction: Let $k>3$, and assume that every polygon with fewer than k vertices has a triangulation.

[Show that polygons with k vertices have triangulations.]



Let P be a polygon with k vertices.

By the lemma, P has a diagonal, which splits P into polygons Q and R. Then each of Q and R has fewer than k vertices.

and R has fewer than k vertices.

By the inductive hypothesis, Q and R have triangulations. Combine these triangulations along with the diagonal of P to obtain a triangulation of P.

Theorem: Every triangulation of a polygon P with n vertices has $n-2$ triangles and $n-3$ diagonals.

Proof (by induction):

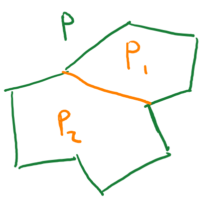
First state the base case. Explain why it is true.

If $n=3$, then P is a triangle. A triangulation of a triangle has $n-2=1$ triangles and
$n-3=0$ diagonals.

Inductive hypothesis: Let $n>3$ be an integer, and assume the statement is true for all polygons with fewer than n vertices.

Now explain how the statement follows for a polygon with n vertices.

If P has n vertices, then choose a diagonal (by the lemma) that splits P into two polygons: P_1 with n_1 vertices and P_2 with n_2 vertices.



Then $n_1+n_2=n+2$.

By the induction hypothesis:

P_1 has n_1-2 triangles and n_1-3 diagonals

P_2 has n_2-2 triangles and n_2-3 diagonals.

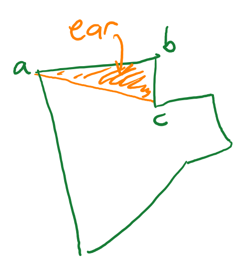
Then the number of triangles in P is:

$$(n_1-2)+(n_2-2)=n_1+n_2-4=n+2-4=n-2.$$

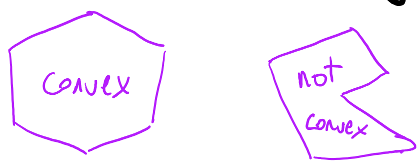
The number of diagonals in P is:

$$(n_1-3)+(n_2-3)+1=n_1+n_2-5=n+2-5=n-3.$$

Note: If a,b,c are consecutive vertices and ac is a diagonal, then abc is called an ear.



CONVEX POLYGON: All diagonals are interior

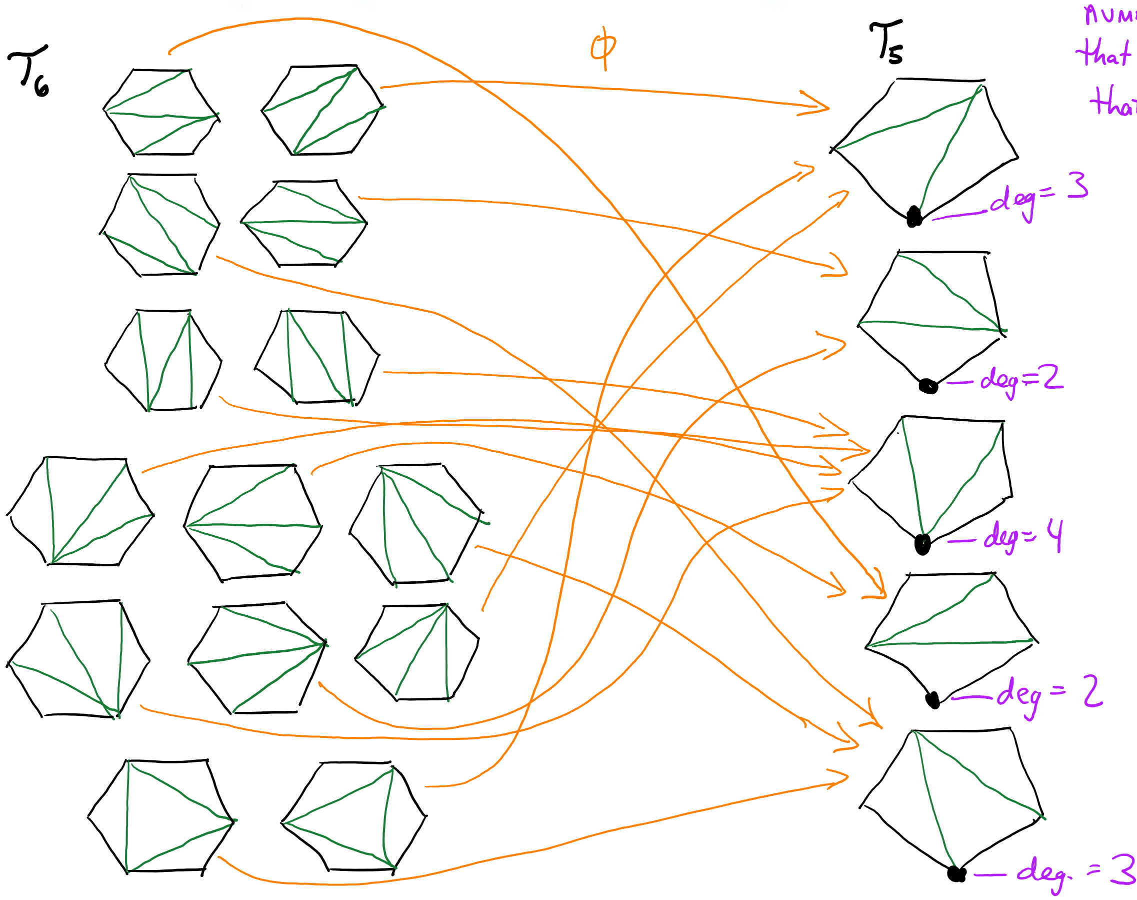


How many triangulations of a convex n-gon?

first consider the following example:

Let P_n be a convex polygon with vertices labeled 1 to n counterclockwise. Let $\mathcal{T}_n$ be the set of all triangulations of P_n, and let t_n be the number of elements of $\mathcal{T}_n$.
Define the map $\phi : \mathcal{T}_6 \to \mathcal{T}_5$ that contracts the edge $\{1, 6\}$ to the point 1. To illustrate this map, first draw all triangulations in $\mathcal{T}_6$ and $\mathcal{T}_5$. Then draw an arrow from each $T \in \mathcal{T}_6$ to $\phi(T)$.

degree of a vertex: the number of edges that meet at that vertex



to be continued...