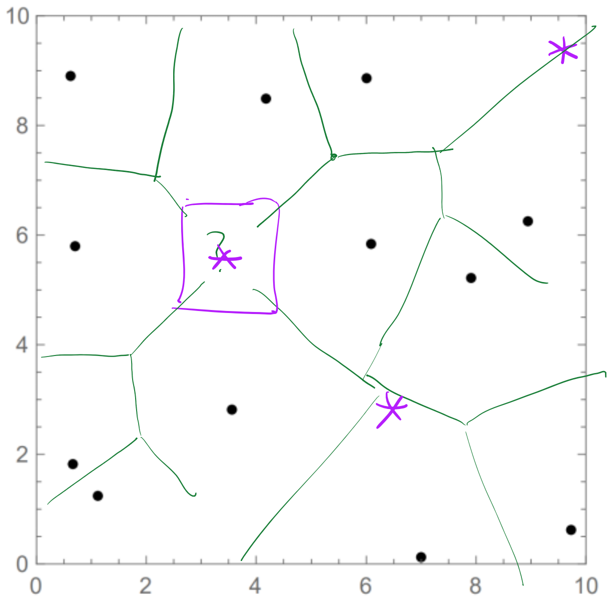
WORKSHEET:



VORONOI DIAGRAM

Let S be a set of points, called "sites" in the plane.

For each site p, the Voronoi region, denoted $Vor(p)$, is the set of points that are at least as close to p as to any other site.

$$Vor(p) = \{ x \in \mathbb{R}^2 \mid \underbrace{|x-p| \le |x-q|}_{\text{absolute values}} \text{ for all sites } q \in S \}$$

such that

The Voronoi diagram $Vor(S)$ is the collection of boundaries of all Voronoi regions. This consists of Voronoi edges and Voronoi vertices.

PROPERTIES OF VORONOI DIAGRAMS

1. All Voronoi regions are convex.
2. Voronoi edges are either line segments, rays, or infinite lines (in both directions).

3. Voronoi vertices are equidistant from their closest sites.

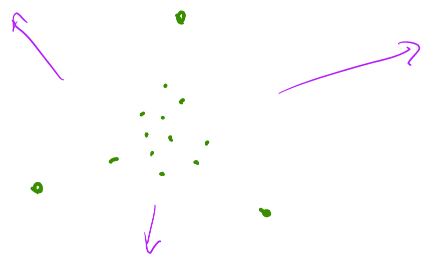
Max numbers of Voronoi vertices and edges?

- $2(n-3)+1$ vertices?

conjectures $2n-5$

- triangular hull?

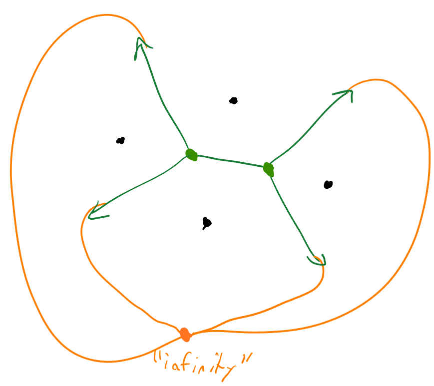
- $3n-6$ edges?



THEOREM: Let S be a planar point set of $n \geq 3$ sites. Then $\mathrm{Vor}(S)$ has at most $2n-5$ Voronoi vertices and $3n-6$ Voronoi edges.

proof: (assume not all sites are collinear)

Introduce a new vertex "at infinity" and connect all infinite edges to that vertex.



This forms a graph G.

- If v is the number of Voronoi vertices, then G has $v+1$ vertices.
- If e is the number of Voronoi edges, then G has e edges also.
- Also G has n faces (one per site)

Euler's formula: $V - E + F = 2$

for G: $(v+1) - e + n = 2 \implies v+1 = 2+e-n$

$\implies e = (v+1)+n-2$

Total degree of G: adding up degrees of all vertices counts every edge twice, and each vertex has degree at least 3

$$\underbrace{2e}_{\text{total degree is } 2e, \text{ which is at least } 3\times(\text{num. vertices})} \geq 3(v+1)$$

Substitute $v+1 = 2+e-n$: $2e \geq 3(2+e-n)$

$2e \geq 6+3e-3n$

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Max number of Voronoi edges $\longrightarrow$ $\boxed{3n-6 \geq e}$

Substitute $e = (v+1)+n-2$: $\quad 2((v+1)+n-2) \geq 3(v+1)$

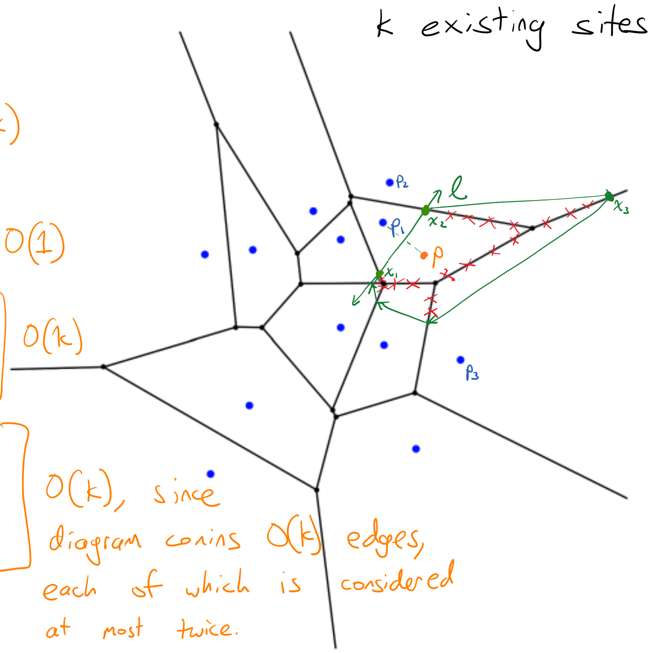
$$2v+2+2n-4 \geq 3v+3$$

Max number of Voronoi vertices $\longrightarrow$ $\boxed{2n-5 \geq v}$

QUESTION: Given a set of points (sites), how do we compute a Voronoi diagram?

Let p be the new site

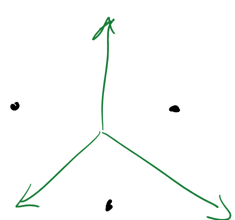
1. Find the existing site closest to p. $\quad O(k)$
2. Calculate the perpendicular bisector ℓ to the line segment $\overline{pp_1}$. $\quad O(1)$
3. Iterate over edges of $Vor(p_1)$ and find intersection points x_1 and x_2 with ℓ. $\quad O(k)$
4. Continue from x_2, constructing $Vor(p)$ segment by segment, until it closes back at x_1. $\quad O(k)$, since diagram contains $O(k)$ edges, each of which is considered at most twice.
5. Remove edges (and vertices) interior to the new region. $\quad O(k)$



INCREMENTAL ALGORITHM:

INPUT: list of n sites.

Construct diagram for the first 3 sites



For each site $4, \dots, n$:
insert new Voronoi region as described above $\quad O(n)$

Overall complexity: $O(n^2)$

DATA STRUCTURE FOR STORING GRAPHS:

DOUBLY-CONNECTED EDGE LIST (DCEL)

define classes: vertex, halfedge, face

vertex: coordinates
adjacent halfedge

halfedge: points to its dual
points to adjacent face
points forward and back

face: pointers to adjacent halfedges