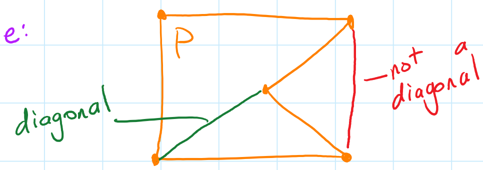
Let P be a polygon.

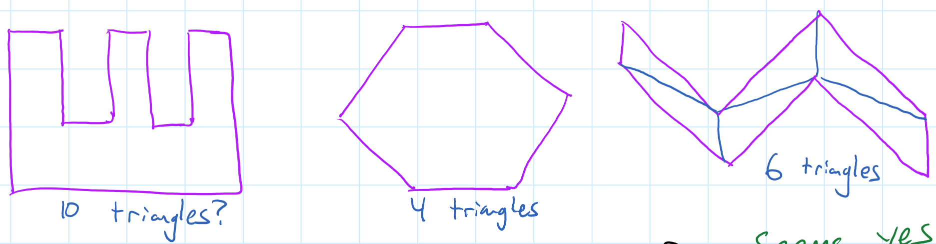
A **DIAGONAL** of P is a line segment connecting two vertices of P, lying inside of P, and only touching the boundary of P at its endpoints.

example:



A **TRIANGULATION** of P is a decomposition of P into triangles by a $\underbrace{\text{maximal set}}_{\text{can't add any more}}$ of noncrossing diagonals.

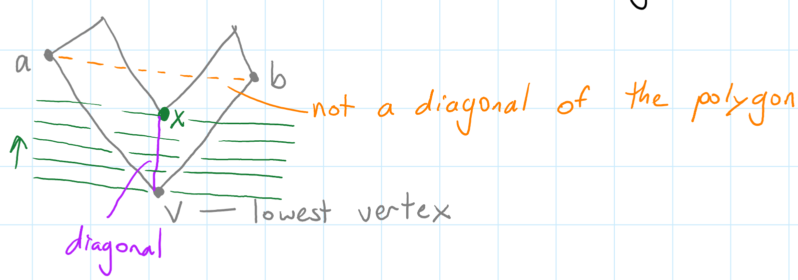
EXAMPLES:



- Does each polygon have a triangulation? — Seems yes
- Is the triangulation unique? If not, how many triangulations are there?
  No.
- How many triangles are in each triangulation?

LEMMA: Every polygon with more than 3 vertices has a diagonal.

Why? line sweep argument

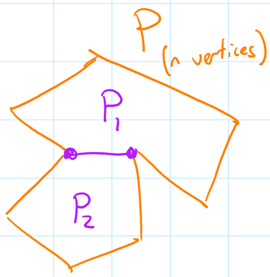


THEOREM: Every polygon has a triangulation.

Proof: Proof by induction on the number of vertices of polygon P.

<u>Base case</u>: Suppose P has 3 vertices.
Then P is a triangle, so we are done.

<u>Induction</u>: Let $n > 3$, and suppose the theorem holds for polygons with less than n vertices.



By the lemma, polygon P has a diagonal, which cuts polygon P into two polygons P_1 and P_2.

Then P_1 and P_2 each have less than n vertices, so they have triangulations.

Since P_1 and P_2 are disjoint (by Jordan Curve Thm), then combine their triangulations to obtain a triangulation of P.

QUESTION: If P has n vertices, how many triangles are in a triangulation of P?

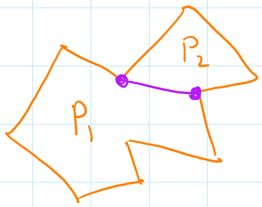
ANSWER: $n-2$ triangles

PROOF: induction!

<u>Base Case</u>: If $n=3$, then P is a triangle, and $1 = 3-2$.

<u>Induction</u>: Suppose $n > 3$ and the statement holds for polygons with less than n vertices.

Choose a diagonal that cuts P into P_1 and P_2.



Let n_1 be the number of vertices in P_1, and

n_2 the number of vertices in P_2.

Then: $n_1 + n_2 = n + 2$ (since endpoints of diagonal are in both P_1 and P_2)

By induction hypothesis,

P_1 has $n_1 - 2$ triangles and P_2 has $n_2 - 2$ triangles.

Then the number of triangles in P is:

$$(n_1 - 2) + (n_2 - 2) = n_1 + n_2 - 4 = n + 2 - 4 = \boxed{n - 2}.$$