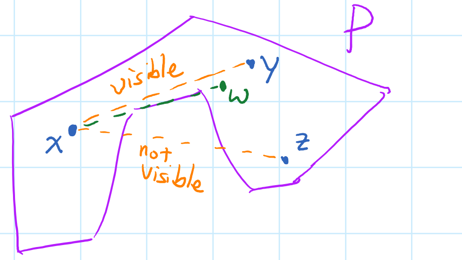
A point x in a polygon P is **VISIBLE** to a point y if the line segment xy lies in P.

Note that the segment may intersect the boundary ∂P.

boundary

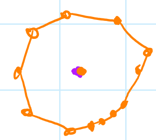


x is visible to w, even though the segment xw includes an edge of P

A set of guards (i.e. points) **COVERS** the polygon if every point in the polygon is visible to some guard.

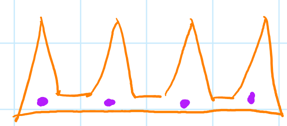
ART GALLERY PROBLEM: Determine the minimum number of guards sufficient to cover any n-walled art gallery (i.e. polygon with n vertices).

example: $n=12$



convex

1 guard suffices



needs 4 guards

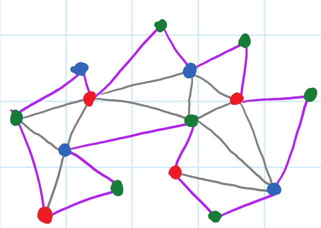
- First proposed Victor Klee in 1973
- Spurred a lot of research. Joseph O'Rourke published a book on Art Gallery Theorems in 1986.

We think that $\lfloor \frac{n}{3} \rfloor$ guards should be sufficient.

"floor" function: greatest integer $\leq \frac{n}{3}$

Proof: Let polygon P have n vertices.

Consider a triangulation of P.



n vertices

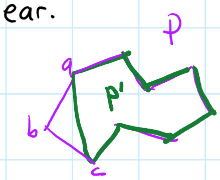
The vertices may be 3-colored so that any two vertices connected by an edge of P or a diagonal in the so that any two vertices connected by an edge of P or a diagonal in the triangulation have different colors.

Proof by induction: <u>Base Case</u>: $n = 3$ vertices

Just color 1 vertex each color.

<u>Induction</u>: Assume that any polygon/triangulation with $n-1$ vertices can be 3-colored. $(n > 3)$

Since $n > 3$, we know the polygon has an ear. Call the ear abc, with b the tip.



Removing the ear produces a polygon P' with $n-1$ vertices. (Vertex b removed.)

By induction hypothesis, P' can be 3-colored.

Then color vertex b differently from vertices a and c.

Thus, polygon P can be 3-colored.

Place guards on the vertices colored with the least-used color.

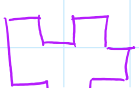
This color has at most $\lfloor \frac{n}{3} \rfloor$ vertices. (If it had more, then each color would have more than $\lfloor \frac{n}{3} \rfloor$ vertices, which would requre more than n vertices total.)

Thus, every triangle is covered by guards, and so is all of P. ■

PROBLEMS:

1. Find a polygon P and a placement of guards such that the boundary ∂P is covered, but not every point of P is covered.

2. What is the minimum number of guards sufficient to cover any polygon with n vertices and all right angles?

"orthogonal polygons" 

"orthogonal polygons"