HW1 regrade:

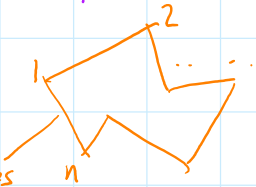
You may resubmit solutions to #1.5, 1.10, 1.15 for a regrade.
Due Monday, Feb. 25. Turn in original and revision.

Notes: For 1.5, induction on the number of holes.
If you want a diagonal between a hole vertex and an exterior vertex, explain how you know such a diagonal exists.

For 1.10, induction on the number of vertices is a good idea.

For 1.15, the formula: $\text{Area} = \frac{1}{2}\left| \sum (x_i y_{i-1} - x_{i-1} y_i) \right|$

absolute value is outside the sum



contributes $x_1 y_n - x_n y_1$ to the sum

SCISSORS CONGRUENCE

Last time: Any two polygons of the same area are scissors congruent. (1830s)

Does a similar theorem hold in 3D?

If two polyhedra have equal volume, are they scissors congruent?

- Hilbert's Third Problem (1900)

3. The Equality of the Volumes of Two Tetrahedra of Equal Bases and Equal Altitudes.

In two letters to Gerling, Gauss * expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, *i. e.*, in modern phraseology, upon the axiom of continuity (or upon the axiom of Archimedes). Gauss mentions in particular the theorem of Euclid, that triangular pyramids of equal altitudes are to each other as their bases. Now the analogous problem in the plane has been solved.† Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts. Nevertheless, it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility. This would be obtained, as soon as we succeeded in *specifying two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra.*‡

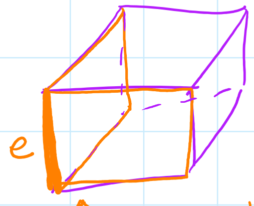
He wants an example of two tetrahedra of same volume that are not scissors congruent.

- Solved by Dehn (1903)
- Also proposed by Kretkowski in 1882, solved by Birkenmajer (work rediscovered later)

DIHEDRAL ANGLE: interior angle between two faces of a polyhedron that meet along an edge.

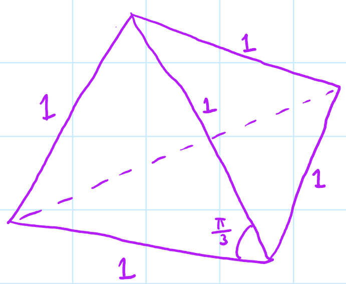
examples:

cube



notation: $\phi(e) = \frac{\pi}{2}$ or $90°$

regular tetrahedron



dihedral angle is $\arccos\left(\frac{1}{3}\right)$ or about $70.53°$

NOTE: $\arccos\left(\frac{1}{3}\right)$ is not of the form $\frac{a}{b}\pi$ for integers a, b

DEHN INVARIANT:

Let $f: \mathbb{R} \to \mathbb{Q}$ be such that:

(a) $f(x+y) = f(x) + f(y)$ for any $x, y \in \mathbb{R}$

(b) $f(qx) = qf(x)$ for any $x \in \mathbb{R}$, $q \in \mathbb{Q}$

(c) $f(\pi) = 0$

"dihedral function" or "d-function"

NOTE: $f(\theta) = 0$ if θ is any rational multiple of π

$$f\left(\tfrac{a}{b}\pi\right) = \tfrac{a}{b} f(\pi) = \tfrac{a}{b} 0 = 0$$

For an edge e of polyhedron P, let $\ell(e)$ be the length of e, and let $\phi(e)$ be the dihedral angle along e.

Call $\ell(e) \cdot f(\phi(e))$ the **MASS** of edge e.

DEHN INVARIANT: for a polyhedron P and d-function f,

$$D_f(P) = \underbrace{\sum_{e \in P} \ell(e) \cdot f(\phi(e))}_{\text{sum of masses of all edges of } P \text{ using d-function } f}$$

example: P is a cube: $D_f(P) = \sum_{e \in P} \ell(e) \cdot f(\phi(e)) = 0$

$f\left(\frac{\pi}{2}\right) = 0$