LAST TIME: scissors congruence of polyhedra

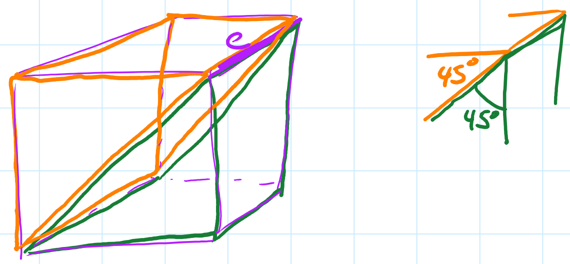
Dehn invariant of polyhedron P: $D_f(P) = \sum_{e \in P} \ell(e) \cdot f(\phi(e))$

THEOREM: The Dehn invariant is invariant under dissections.

That means: If P is dissected into polyhedra $P_1, P_2, \ldots, P_n$,

then: $$D_f(P) = D_f(P_1) + D_f(P_2) + \cdots + D_f(P_n)$$

sketch of proof: relies on the fact that dihedral functions are additive: $f(\phi_1 + \phi_2) = f(\phi_1) + f(\phi_2)$



edge e contributes to the Dehn invariant:

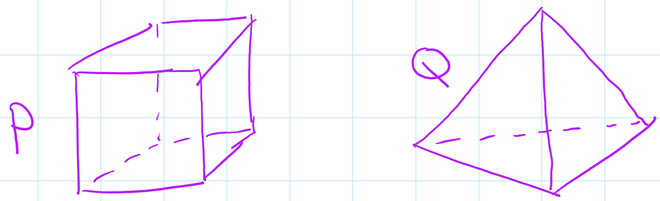
$$\ell(e) \cdot f(90^\circ) = \ell(e)\left(f(45^\circ) + f(45^\circ)\right)$$

Any new edge through a face of P contributes zero, since all dihedral angles of such edge sum to $\frac{\pi}{2}$.

Any new edge in the interior of P also contributes zero, since its dihedral angles sum to π.

COROLLARY: Let P and Q be polyhedra, and f any dihedral function. If $D_f(P) \neq D_f(Q)$, then P and Q are not scissors congruent.

What does this tell us about the cube and regular tetrahedron?



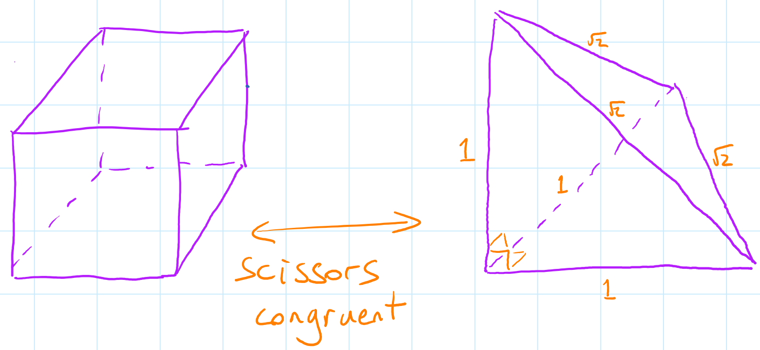
$D_f(P) = 0$ $\qquad$ $D_f(Q) \neq 0$ if $f(\underbrace{\arccos(\tfrac{1}{3})}_{\text{not a rational multiple of } \pi}) \neq 0$

They are not scissors congruent!

THEOREM (Sydler, 1965): Polyhedra P and Q are scissors congruent if they have the same volume and the same Dehn invariant:

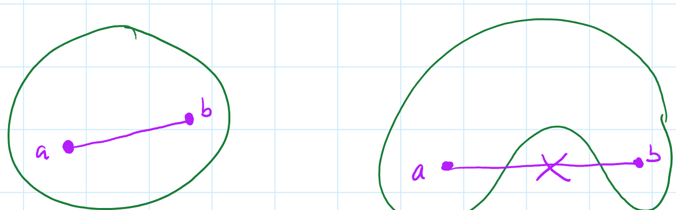
$$D_f(P) = D_f(Q) \quad \text{for } \underline{\text{every}} \text{ } d\text{-function } f.$$

example:



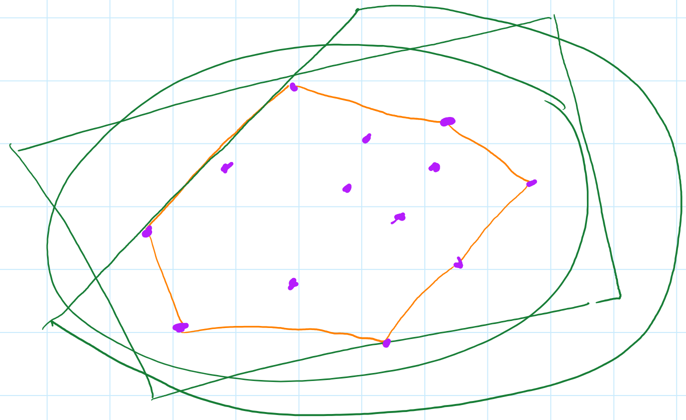
CONVEXITY:

A region R is **CONVEX** if, for any two points a and b in R, the line segment ab is also in R.



Convex $\qquad$ not convex

Let S be a set of points. The **CONVEX HULL** of S, denoted $\mathrm{conv}(S)$, is the intersection of all convex regions containing S.



APPLICATIONS:

- collision detection
- robot motion planning
- geographic information systems
- image processing
- optimization (simplex algorithm)
- spread of epidemics
- cooking

QUESTION: Given coordinates of a set of points S in the plane, how would you program a computer to find the convex hull $\mathrm{conv}(S)$?