EULER CHARACTERISTIC: ANOTHER PERSPECTIVE

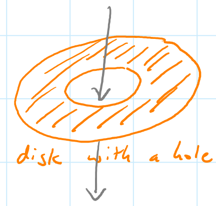
Euler characteristic can also be computed in terms of the Betti numbers of a surface or polyhedron.

β_0 = number of connected components

β_1 = number of holes (1-D holes)

β_2 = number of voids (2-D holes)

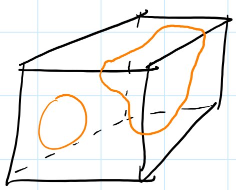
⋮ e.g. space inside of a balloon.



$$\boxed{\chi = \beta_0 - \beta_1 + \beta_2}$$

EXAMPLES:

(1)



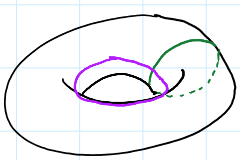
$\beta_0 = 1$ (cube is connected)

$\beta_1 = 0$ (any loop on the surface is contractible to a point)

$\beta_2 = 1$ (void inside of the cube)

$$\beta_0 - \beta_1 + \beta_2 = 1 - 0 + 1 = 2 = \chi(\;)$$

(2)



$\beta_0 = 1$ (torus is connected)

$\beta_1 = 2$ (two different types of loops)

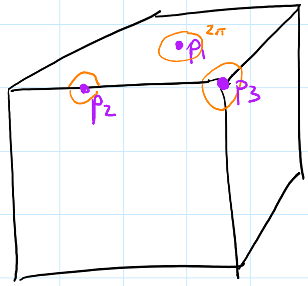
$\beta_2 = 1$ (interior is a void)

$$\beta_0 - \beta_1 + \beta_2 = 1 - 2 + 1 = 0 = \chi(\;)$$

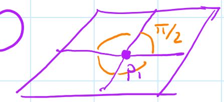
CURVATURE

The curvature $K(p)$ at a point p on a polyhedron is 2π minus the sum of face angles at p.

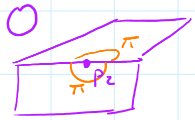
EXAMPLE:



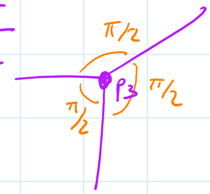
$K(p_1) = 2\pi - 2\pi = 0$



$K(p_2) = 2\pi - 2\pi = 0$



$K(p_3) = 2\pi - \frac{3\pi}{2} = \frac{\pi}{2}$



Nonzero curvature only occurs at the corners of the cube.

Total curvature: 8 corners, each with curvature $\frac{\pi}{2}$

$$\sum_{\substack{v \in P \\ \text{vertex}}} K(v) = 8\left(\frac{\pi}{2}\right) = 4\pi$$

MORE EXAMPLES:

TETRAHEDRON: 4 triangular faces

each vertex: $3\left(\frac{\pi}{3}\right) = \pi$
4 vertices total,
so total curvature is 4π

OCTAHEDRON: 8 triangular faces

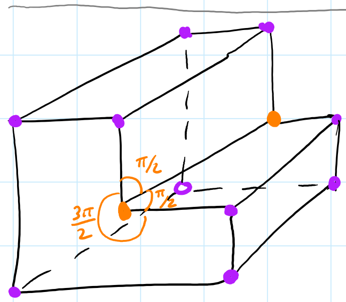
each vertex: $\frac{2\pi}{3}$
6 vertices total, so total curvature is 4π

DODECAHEDRON: 12 pentagonal faces

each vertex $\frac{\pi}{5}$
20 vertices, so total curvature is 4π

ICOSAHEDRON: 20 triangular faces

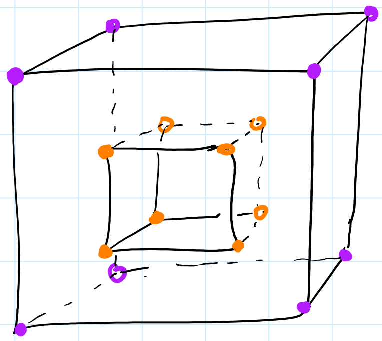
each vertex $\frac{\pi}{3}$
12 vertices, so total curvature is 4π



purple vertices: $2\pi - 3\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$
10 total

orange vertices: $2\pi - \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{3\pi}{2}\right) = -\frac{\pi}{2}$
2 total

Total curvature: $10\left(\frac{\pi}{2}\right) + 2\left(\frac{-\pi}{2}\right) = 5\pi - \pi = \boxed{4\pi}$



purple vertices: $\frac{\pi}{2}$ (8 total)

orange vertices: $\frac{-\pi}{2}$ (8 total)

Total curvature: $8\left(\frac{\pi}{2}\right) + 8\left(\frac{-\pi}{2}\right) = 0$

POLYHEDRAL GAUSS-BONNET THEOREM

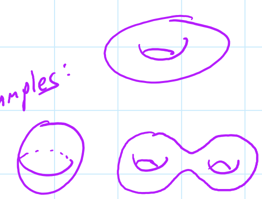
For a polyhedron P, the total curvature is $2\pi\,\chi(P)$.

That is,

$$\underbrace{\sum_{\substack{v\in P\\ \text{(vertices)}}} K(v)}_{\substack{\text{Curvature}\\ \text{(geometry)}}} = 2\pi\,\underbrace{\chi(P)}_{\substack{\text{Euler characteristic}\\ \text{(topology)}}}.$$

FOR SURFACES: GAUSS-BONNET THEOREM

examples:



Let S be a smooth surface without boundary.

Then $\int_S K\, dA = 2\pi\,\chi(S)$.

Gaussian curvature
(differential geometry)
integrate over the surface