

Exam 1

Math 330

Due Tuesday, October 10 at **8am**

Name: _____

Instructions:

- Solve any 5 of the following 6 problems.
- You may use your textbook, your notes, the course web site, *Mathematica*, *Wolfram Alpha*, and homework assignments/solutions.
- *Do not consult other sources, web sites, or people other than the professor.*
- Type your solutions in L^AT_EX. If you use technology to compute something, indicate what you computed. Make sure to explain your solutions clearly, check your work, and proofread.
- *Make sure you attend to the pledge that at the end of this exam.*

1. Consider the following simple 1-D model for traffic flow in the interval from $x = a$ to $x = b$. Let x be the position on a number line and t be the time. We define two functions: $\rho(x, t)$ is the density of cars (cars per unit length) at position x and time t and $v(x, t)$ is the velocity of the cars at position x and time t . Use the convention that a positive velocity implies cars are moving in the direction from a to b . Assume cars can only enter/exit this region at the endpoints $x = a$ and $x = b$. Write a conservation equation and derive the corresponding partial differential equation.

2. Suppose $u(x, t)$ is the temperature of a rod for $0 < x < L$ and $t > 0$, subject to

$$\begin{aligned} \text{PDE:} & \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x - \beta \\ \text{Boundary conditions:} & \quad \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0 \\ \text{Initial condition:} & \quad u(x, 0) = f(x) \end{aligned}$$

- Calculate the value of the constant β such that an equilibrium temperature exists. Briefly explain why this value of β makes sense based on the physical context.
- Using the β value found in part (a), determine the equilibrium temperature in terms of the initial condition.

3. Consider a circular annulus defined on $a < r < b$, $-\pi < \theta < \pi$. Solve Laplace's equation inside the annulus subject to the boundary conditions

$$\frac{\partial u}{\partial r}(a, \theta) = 0, \quad u(b, \theta) = g(\theta).$$

Recall, in polar coordinates,

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

4. Consider a convective-diffusion equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial u}{\partial x} + 8u, & t > 0, x \in (0, L), \\ u(t, 0) &= 0, u(t, L) = 0, \\ u(0, x) &= u_0(x), & x \in (0, L). \end{aligned}$$

Recall in Exercise 1.5.2 you considered convection as another process that introduces flux. The term $4u_x$ is the convective term in this problem. Here, you will find the solution of the equation with the following steps:

- (a) Use separation of variables method to show that if $u(t, x) = G(t)\phi(x)$ is a solution, then for some constant λ , G and ϕ satisfy

$$G' = \lambda G, \quad \phi'' + 4\phi' + 8\phi = \lambda\phi.$$

- (b) Find the eigenvalues and eigenfunctions of

$$\phi'' + 4\phi' + 8\phi = \lambda\phi, \quad \phi(0) = \phi(L) = 0.$$

(You will need to consider three cases for λ separately.)

- (c) Find the solution of the equation in a series form. Make sure that you write your coefficients in terms of the initial condition.

5. Find $u(x, t)$ that solves the following:

PDE:
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - 4u, \quad 0 < x < 1, t > 0$$

Boundary conditions:
$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t) = 0$$

Initial condition:
$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 5 + 3 \cos(2\pi x)$$

6. Solve Laplace's equation inside a rectangle $0 < x < L$, $0 < y < H$, with the boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, H) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}.$$

St. Olaf Honor Pledge: I pledge my honor that on this examination I have neither given nor received assistance not explicitly approved by the professor and that I have seen no dishonest work.

Signed: _____

I have intentionally not signed the pledge. (Check the box if appropriate.)