Exam 1 Name: ______________________________

Math 330

Due Tuesday, October 9 at **11:45am**

Instructions:

- Solve any 5 of the following 6 problems.
- You may use your textbook, your notes, the course web site, computational technology (e.g., *Mathematica*), and homework assignments/solutions.
- *Do not consult other sources, web sites, or people other than the professor.*
- Type your solutions in LaTeX. If you use technology to compute something, indicate what you computed. Make sure to explain your solutions clearly, check your work, and proofread.
- *Make sure you attend to the pledge that at the end of this exam.*

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1. Consider the following simple 1-D model for traffic flow in the interval from $x = a$ to $x = b$. Let x be the position on a number line and t be the time. We define two functions: $\rho(x, t)$ is the density of cars (cars per unit length) at position x and time t and $v(x, t)$ is the velocity of the cars at position x and time t. Use the convention that a positive velocity implies cars are moving in the direction from a to b. Assume cars can only enter/exit this region at the endpoints $x = a$ and $x = b$. Write a conservation equation and derive the corresponding partial differential equation.

2. Suppose $u(x, t)$ is the temperature of a rod for $0 < x < L$ and $t > 0$, subject to

PDE:	$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + x - \beta$
Boundary conditions:	$\dfrac{\partial u}{\partial x}(0, t) = \dfrac{\partial u}{\partial x}(L, t) = 0$
Initial condition:	$u(x, 0) = f(x)$

(a) Calculate the value of the constant β such that an equilibrium temperature exists. Briefly explain why this value of β makes sense based on the physical context.

(b) Using the β value found in part (a), determine the equilibrium temperature function in terms of the initial condition.

3. Consider a circular annulus defined on $a < r < b$, $-\pi < \theta < \pi$. Solve Laplace's equation inside the annulus subject to the boundary conditions

$$\frac{\partial u}{\partial r}(a, \theta) = 0, \qquad u(b, \theta) = g(\theta).$$

Recall that Laplace's equation in polar coordinates is:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Hint: Don't forget about section 2.5.2 in the text.

4. Consider a convective-diffusion equation

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + 4\frac{\partial u}{\partial x} + 8u, \qquad t > 0, x \in (0, L), \\
u(t,0) &= 0, u(t,L) = 0, \\
u(0,x) &= u_0(x), \qquad x \in (0, L).
\end{aligned}$$

This is simply the heat equation with extra terms to model convection. Use the following steps to solve the PDE above.

(a) Use separation of variables to show that if $u(t,x) = G(t)\phi(x)$ is a solution, then for some constant λ, G and ϕ satisfy

$$G' = \lambda G, \qquad \phi'' + 4\phi' + 8\phi = \lambda\phi.$$

(b) Find the eigenvalues and eigenfunctions of

$$\phi'' + 4\phi' + 8\phi = \lambda\phi, \qquad \phi(0) = \phi(L) = 0.$$

(You will need to consider three cases for λ separately.)

(c) Find the solution of the equation in a series form. Make sure that you write your coefficients in terms of the initial condition.

5. Find $u(x,t)$ that solves the following:

PDE: $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - 4u, \quad 0 < x < 1, t > 0$$

Boundary conditions: $$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$$

Initial condition: $$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 5 + 3\cos(2\pi x)$$

6. Solve Laplace's equation inside a rectangle $0 < x < L$, $0 < y < H$, with the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad \frac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = 0, \qquad \frac{\partial u}{\partial y}(x,H) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}.$$

St. Olaf Honor Pledge: I pledge my honor that on this examination I have neither given nor received assistance not explicitly approved by the professor and that I have seen no dishonest work.

Signed: ________________________________

☐ I have intentionally not signed the pledge. (Check the box if appropriate.)