Generalized Functions

Math 330

Consider the boundary-value problem

$$-c\frac{d^2u}{dx^2} = f(x), \qquad u(0) = u(1) = 0,$$

which models the deflection of a unit-length elastic bar of stiffness c subject to a force $f(x)$.

1. Let $f(x) = \delta_\xi(x)$ for some $\delta \in (0,1)$. The differential equation is now

$$-c\frac{d^2u}{dx^2} = \delta_\xi(x).$$

Integrate twice to find $u(x)$. Your answer should contain two integration constants.

2. Apply the boundary conditions $u(0) = u(1) = 0$ to solve for your integration constants.

3. Let $G(x;\xi)$ be the solution you found in #2. Sketch the graph of $G(x;\xi)$.

4. Complete the following sentences.

(a) The function $G(x;\xi)$ is called

(b) We interpret $G(x;\xi)$ as

(c) For a general forcing function $f(x)$, the solution to the BVP is given by

5. For a constant unit force $f(x) = 1$ for $0 < x < 1$, find the solution $u(x)$ to the BVP.

Consider the boundary-value problem

$$-\frac{d^2u}{dx^2} + \omega^2 u = f(x), \qquad u(0) = u(1) = 0, \qquad \omega > 0.$$

6. First, let $f(x) = \delta_\xi(x)$ for $0 < \xi < 1$. Show that

$$G(x;\xi) = \begin{cases} a\sinh(\omega x), & x < \xi, \\ b\sinh(\omega(1-x)), & x > \xi \end{cases}$$

satisfies the BVP for $0 \le x < \xi$ and $\xi < x \le 1$.

7. If $G(x;\xi)$ is to be continuous and $\frac{dG}{dx}(x;\xi)$ has a jump discontinuity that matches that of $\sigma_\xi(x)$, what system of equations can you solve for a and b?

8. Solving for a and b yields

$$G(x;\xi) = \begin{cases} \frac{\sinh(\omega(1-\xi))\sinh(\omega x)}{\omega\sinh(\omega)}, & x \le \xi, \\ \frac{\sinh(\omega(1-x))\sinh(\omega\xi)}{\omega\sinh(\omega)}, & x > \xi. \end{cases}$$

Now integrate to find $u(x)$ that solves the BVP with $f(x) = 1$. Plot your solution.