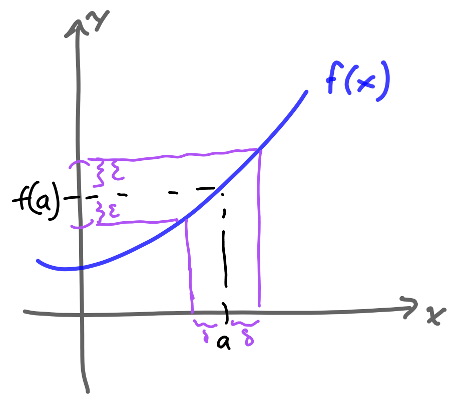
September 10, 2024

Math 348

DEFINITION. Given any subset $S \subset \mathbb{R}$, a function $f: S \to \mathbb{R}$ is continuous at a point $a \in S$ if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$|f(a) - f(x)| < \varepsilon \quad \text{whenever} \quad |x - a| < \delta.$$

A function $f: S \to \mathbb{R}$ is continuous if it is continuous at every point in S.



These definitions are from Real Analysis.

In topology, we want to think about continuity in terms of open sets.

Let's see how to do that...

Open Sets in $\mathbb{R}$

MATH 348

For today, all of the sets we consider will be sets of real numbers.

1. Give an example of a set of real numbers...

(a) ...that is neither open nor closed. — $\mathbb{Q}$ rationals, $[0,1)$

(b) ...that is both open and closed.

empty set: $\emptyset$ is both open and closed

$\mathbb{R}$ is both open and closed

NOTE: $\emptyset$ is open in $\mathbb{R}$ since the definition of open (page 9 in the text) is vacuously true for $\emptyset$.

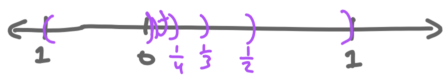
2. Give a proof or counterexample for each statement.

(a) The union of any collection of open sets is itself an open set. — Proposition 2.4 in the text

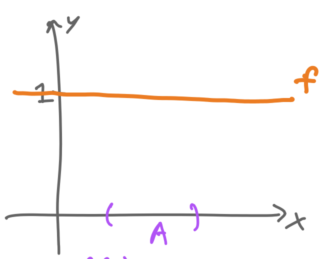
(b) The intersection of any collection of open sets is itself an open set. — False. Here is a counterexample.

(c) The intersection of any finite collection of open sets is itself an open set. proof on next page

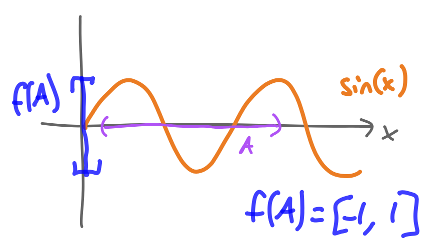
(b) $\bigcap_{i=1}^{\infty} \left(-1, \frac{1}{i}\right) = (-1, 0]$ not open



3. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is open, is $f(A)$ necessarily open? No. Here are some counterexamples.

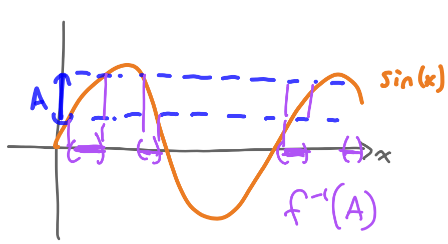


$f(A) = \{1\}$ closed, not open



4. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is open, is $f^{-1}(A)$ necessarily open?

$\underbrace{f^{-1}(A)}$ preimage, inverse image



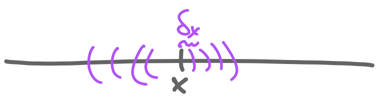
Proof of 2(c):

Let $S_1, S_2, \dots, S_n$ be a finite collection of open sets

Let $x \in \bigcap_{i=1}^{n} S_i$. So x is in each S_i.

Since S_i is open, there exists some interval

$$(x - \delta_{x,i}, \ x + \delta_{x,i}) \subset S_i \qquad (\text{for all } i=1,\dots,n).$$



Since there are finitely many

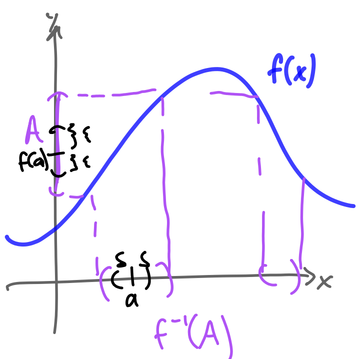
$$\delta_{x,1}, \ \delta_{x,2}, \ \dots, \ \delta_{x,n} \qquad (\text{all positive})$$

there exists $\delta_x = \min_i \{\delta_{x,i}\} > 0$ ← Note that this minimum exists because we have a finite collection of positive values

Thus, $(x - \delta_x, x + \delta_x)$ is contained in all S_i, and thus in the intersection $\bigcap_{i=1}^{n} S_i$, which is therefore open.

THEOREM: If $f: \mathbb{R} \to \mathbb{R}$ is continuous (by the ε-δ definition) and $A \subset \mathbb{R}$ is open, then $f^{-1}(A)$ is open.

NOTE: This theorem is one direction of Theorem 29 in the text. See page 13 in the text for a proof of both directions.



it would have been better to write this first.

proof: Let $a \in f^{-1}(A)$. So $f(a) \in A$.

Since f is continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$

whenever $x \in (a - \delta, a + \delta)$ — def. of continuity

Since A is open and $f(a) \in A$, there exists $\varepsilon > 0$ such that $(f(a) - \varepsilon, f(a) + \varepsilon) \subset A$.

Then, continuity of f gives a δ small enough that $(a - \delta, a + \delta) \subset f^{-1}(A)$.

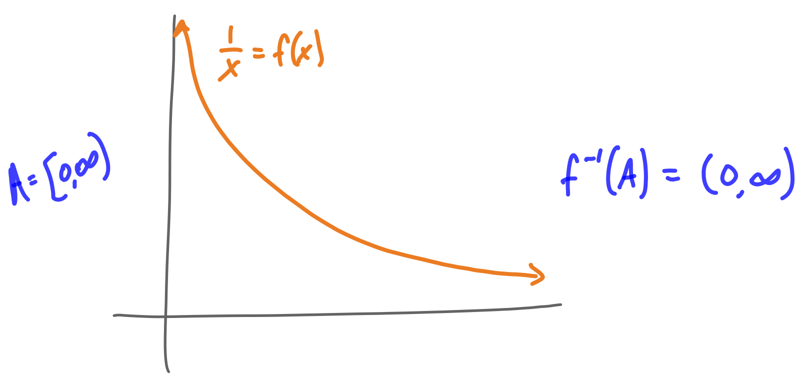
So $a \in (a - \delta, a + \delta) \subset f^{-1}(A)$, which shows that $f^{-1}(A)$ is open.

THEOREM 2.9 in the text:

Function f is continuous if and only if the preimage of every open set is open.

This is a key idea in topology!

scratch work:



5. Can you formulate a statement about continuity in terms of *closed* sets? That is, complete the following statement.

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if ____*something about closed sets*____.

We'll return to this later.

These are bonus problems. Think about them if you want to!

6. Let A and B be open sets and $X = A \cup B$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be continuous functions that agree on $A \cap B$. Define the function $h : A \cup B \to \mathbb{R}$ by $h = f$ on A and $h = g$ on B. Is h continuous on X? Why or why not?

7. How would your answer to #6 change if A and B are *closed* sets?

8. How would your answer to #6 change if A and B are *arbitrary* sets?