MATH 348
12 September 2024

> Definition: A **topological space** is a set, X, together with a collection, $\mathcal{T}$, of subsets of X, called "open" sets, which satisfy the following rules:
>
> T1. The set X itself is "open"
>
> T2. The empty set is "open"
>
> T3. Arbitrary unions of "open" sets are "open"
>
> T4. Finite intersections of "open" sets are "open"

> Definition: A function $f : S \to T$ from one topological space to another is **continuous** if the preimage $f^{-1}(Q)$ of every open set $Q \subset T$ is an open set in S.

If T is any topological space (for example, $\mathbf{R}^n$), and $S \subset T$ is any subset of T, then the **subspace topology**[1] on S is defined as follows: A subset of S is said to be open (in the subspace topology) if it is the intersection of S with an open set in T. With these open sets, S becomes a topological space in its own right, and we refer to S as a **subspace** of T.

Topological Spaces

MATH 348

1. Find all possible topologies on $X = \{a, b\}$.

$\tau_1 = \{\{a,b\}, \emptyset\}$ indiscrete topology

$\tau_2 = \{\{a,b\}, \emptyset, \{a\}\}$ $\tau_4 = \{\{a,b\}, \emptyset, \{b\}\}$

$\tau_3 = \{\{a,b\}, \emptyset, \{a\}, \{b\}\}$ discrete topology

2. Find all topologies on $X = \{a, b, c\}$ that contain $\{a\}$ and $\{b\}$ as open sets.

$\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$

$\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ – discrete topology

$\tau_3 = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$ $\tau_4 = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$

3. Give an example of a topology on $\mathbb{R}$ with...

(a) ...exactly three open sets. $\{\emptyset, \{0\}, \mathbb{R}\}$

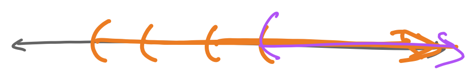
(b) ...exactly five open sets. $\{\emptyset, \{a\}, \{b\}, \{a,b\}, \mathbb{R}\}$

$a, b \in \mathbb{R}$, $a \neq b$

4. Is it possible for a topological space to have exactly one open set?

Yes, if $X = \emptyset$. Then the topology is: $\{\emptyset\}$.

5. Let $X = \mathbb{R}$. Which of the following collections of sets form topologies on X?



(a) $\mathcal{T}_1 = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ Yes!

(b) $\mathcal{T}_2 = \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ No – it doesn't satisfy union property $\bigcup_{i=1}^{\infty} [\frac{1}{i}, \infty) = (0, \infty) \notin \mathcal{T}_2$

(c) $\mathcal{T}_3 = \{S \subset \mathbb{R} \mid 0 \in S\} \cup \{\emptyset\}$ Yes!

(d) $\mathcal{T}_4 = \{S \subset \mathbb{R} \mid S$ contains either 0 or 1$\} \cup \{\emptyset\}$ No. consider $\{0, 3\} \cap \{1, 3\} = \{3\} \notin \mathcal{T}_4$

(e) $\mathcal{T}_5 = \{(a, b) \mid a, b \in \mathbb{R}\}$ No.

$\mathbb{R} \notin \mathcal{T}_5$ also, $(0,1) \cup (2,3) \notin \mathcal{T}_5$

6. Consider the space $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Which of the following functions are continuous in this topology?

(a) $f : X \to X$ defined by $f(a) = b$, $f(b) = c$, and $f(c) = a$ No: $f^{-1}(\{a\}) = \{c\}$ not open!

(b) $g : X \to X$ defined by $g(a) = b$, $g(b) = a$, and $g(c) = c$ $f^{-1}(\{b\}) = \{a\}$ is open

$g^{-1}(\{a\}) = \{b\}$ $g^{-1}(\{a,b,c\}) = \{a,b,c\}$

$g^{-1}(\{b\}) = \{a\}$ $g^{-1}(\emptyset) = \emptyset$

$g^{-1}(\{a,b\}) = \{a,b\}$

So g is continuous in this topology

7. State another continuous function $h : X \to X$ for the topological space in problem #6.

$h(a)=a,\ h(b)=b,\ h(c)=c$ — the identity function

or

$h(x)=a$ for $x \in X$

— class ended here —

8. Consider the topology $\mathcal{T}_1$ in problem #5. Which of the following functions are continuous?

to be continued...

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$

(b) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -x$

9. Give several examples of continuous functions on the topological space $\mathcal{T}_3$ in problem #5.

10. Let X be a topological space, and let $Y \subset X$ have the subspace topology.

(a) If A is open in Y, and Y is open in X, show that A is open in X.

(b) If A is closed in Y, and Y is closed in X, show that A is closed in X.

11. Consider the following subsets of $\mathbb{R}$ with the subspace topology.

(a) Let $K = \left\{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{Z}_+\right\}$. Show that the subspace topology on K is the discrete topology.

(b) Let $K^* = K \cup \{0\}$. Show that the subspace topology on K is not the discrete topology.