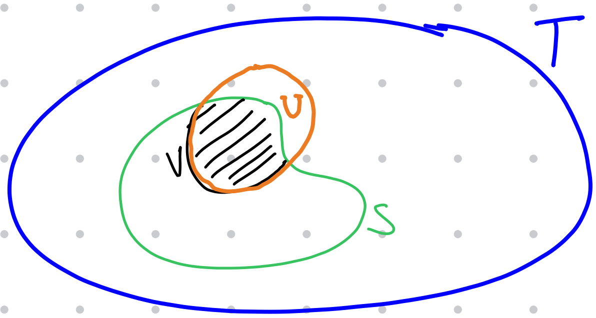
17 Sept. 2024

> Definition: A function $f : S \to T$ from one topological space to another is **continuous** if the preimage $f^{-1}(Q)$ of every open set $Q \subset T$ is an open set in S.

If T is any topological space (for example, $\mathbf{R}^n$), and $S \subset T$ is any subset of T, then the **subspace topology**[1] on S is defined as follows: A subset of S is said to be open (in the subspace topology) if it is the intersection of S with an open set in T. With these open sets, S becomes a topological space in its own right, and we refer to S as a **subspace** of T.

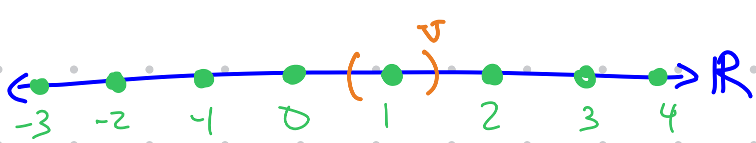


If U is open in T, and $V = U \cap S$, then V is open in S.

EXAMPLES:

- Integers $\mathbb{Z}$ are a subset of the Reals $\mathbb{R}$ with the standard topology

  Subspace topology on $\mathbb{Z}$:



$U \cap \{1\} = \{1\}$

Since U is open in $\mathbb{R}$, $\{1\}$ is open in the subspace topology on $\mathbb{Z}$

Similarly, $\{n\}$ is open for all integers n.

Subspace topology on $\mathbb{Z}$ is the discrete topology.

7. State another continuous function $h : X \to X$ for the topological space in problem #6.

interval $(-\infty, \infty) = \mathbb{R}$

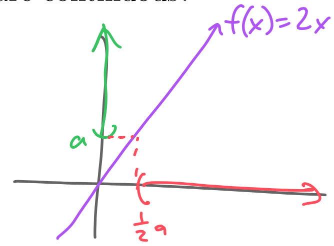
$(-\infty, \infty) \subset \mathbb{R}$

$\mathbb{R} \subset (-\infty, \infty)$

$\mathcal{T}_1 = \{ (a, \infty) \mid a \in \mathbb{R} \} \cup \{\emptyset, \mathbb{R}\}$

8. Consider the topology $\mathcal{T}_1$ in problem #5. Which of the following functions are continuous?

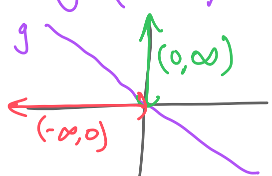
(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ YES: $f^{-1}\big((a,\infty)\big) = \left(\tfrac{1}{2}a, \infty\right)$



(b) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -x$

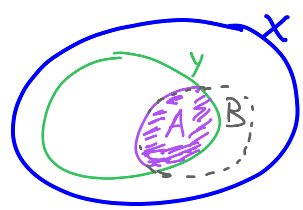
NO: counterexample

Consider $g^{-1}\big((0,\infty)\big) = (-\infty, 0)$



9. Give several examples of continuous functions on the topological space $\mathcal{T}_3$ in problem #5.

10. Let X be a topological space, and let $Y \subset X$ have the subspace topology.



(a) If A is open in Y, and Y is open in X, show that A is open in X.

(b) If A is closed in Y, and Y is closed in X, show that A is closed in X.

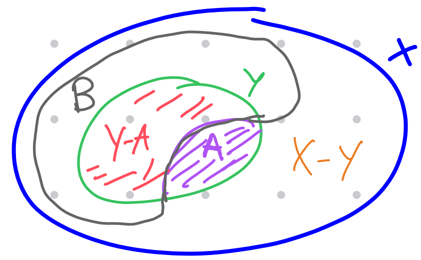
(a) Since A is open in Y, $A = Y \cap B$ for some open set B in X.
Also Y is open in X, so $Y \cap B$ is open in X
But $A = Y \cap B$, so A is open in X.

11. Consider the following subsets of $\mathbb{R}$ with the subspace topology.

(a) Let $K = \{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{Z}_+\}$. Show that the subspace topology on K is the discrete topology.

(b) Let $K^* = K \cup \{0\}$. Show that the subspace topology on K is not the discrete topology.

⑩ (b) Since $A \subset Y$ is closed,
its complement in Y, $Y - A$,
is open in Y.



So $Y - A$ is the intersection of Y with some open set B in X.

That is, $Y - A = B \cap Y$.

Since Y is closed in X, $X - Y$ is open in X.

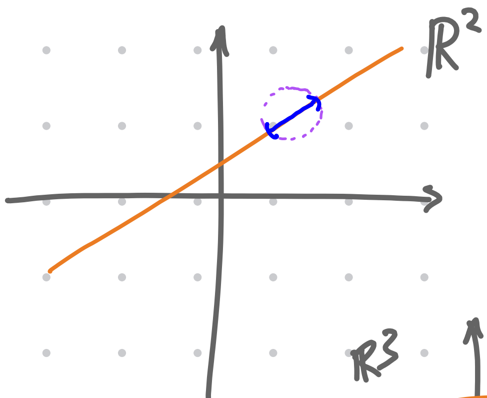
The union $B \cup (X - Y)$ is thus open in X.

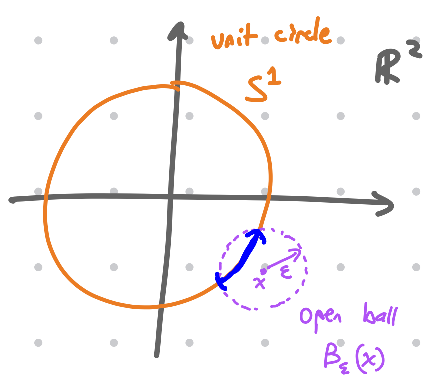
Also, $X - A = B \cup (X - Y)$,

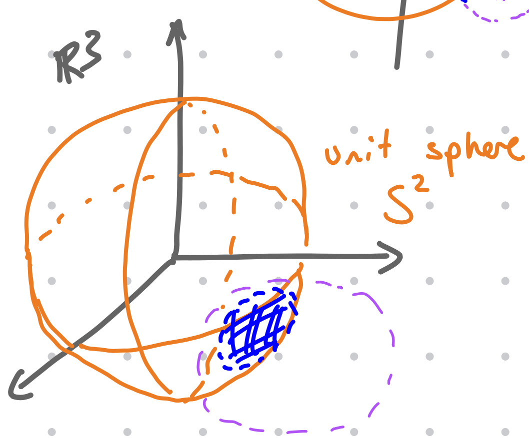
so $X - A$ is open

Thus, A is closed in X.

- Any subset of $\mathbb{R}^n$ inherits a topology from the standard topology on $\mathbb{R}^n$.







- Matrix spaces: $GL(3,\mathbb{R})$ is space of invertible 3x3 matrices with real coefficients

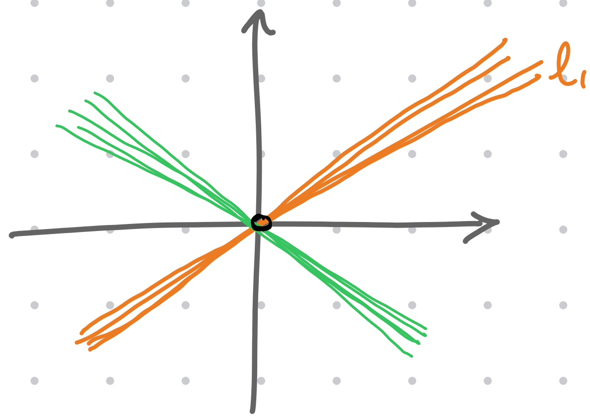
$$GL(3,\mathbb{R}) \subset \mathbb{R}^9$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Real Projective Spaces:

  $\mathbb{RP}^1$ is space of lines through the origin in $\mathbb{R}^2$

  $\mathbb{RP}^n$ " " " " " " " " $\mathbb{R}^{n+1}$



Proposition 3.35:

Let S and T be topological spaces and $f : S \to T$ a continuous map. Suppose that $Q \subset S$ is a subset whose image, under f, is contained in a subset $R \subset T$, so that f can be restricted to a function $f|_Q : Q \to R$. If Q and R are given the subspace topologies, then $f|_Q$ is continuous.

subset

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \cup & & \cup \\ Q & \xrightarrow{f|_Q} & R \end{array}$$

$S \supset Q$

$Q \subset S$