19 September 2024

Continuity (in the subspace topology)

Define the addition map $A: \mathbb{R}^2 \to \mathbb{R}$ by

$$A(x,y) = x+y.$$

Show A is continuous (in the standard topology).

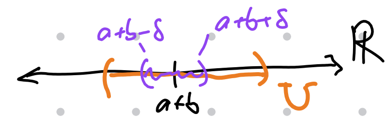
Let $U \subset \mathbb{R}$ be open. Want to show $A^{-1}(U)$ is open.

Suppose $(a,b) \in A^{-1}(U)$, means that $A(a,b) = a+b \in U$.

point in $\mathbb{R}^2$

Since U is open, there exists some $\delta > 0$ such that

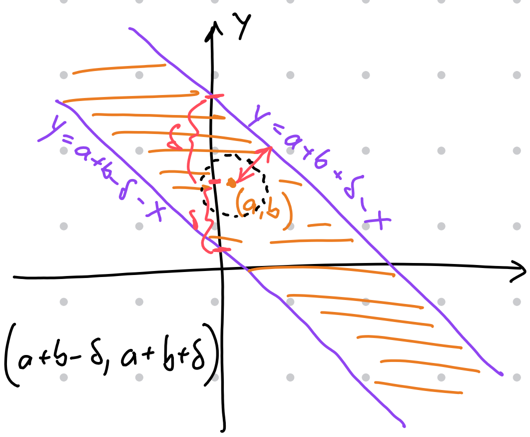
$$(a+b-\delta,\ a+b+\delta) \subset U.$$



Consider $\underbrace{A^{-1}(a+b-\delta,\ a+b+\delta)}_{\text{set of pairs } (x,y) \text{ whose sum is in } (a+b-\delta,\ a+b+\delta)}$

$$\underbrace{a+b-\delta < x+y}\ \underbrace{<\ a+b+\delta}$$

$a+b-\delta = x+y$	$x+y = a+b+\delta$
$y = a+b-\delta-x$	$y = a+b+\delta-x$



Therefore, there exists $B_\varepsilon(a,b) \subset A^{-1}(a+b-\delta,\ a+b+\delta)$

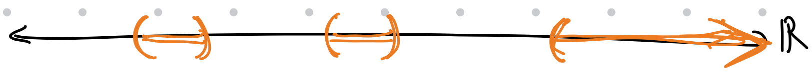
specifically choose $0 < \varepsilon < \frac{\delta}{\sqrt{2}}$.

So thus, $A^{-1}(U)$ is open.

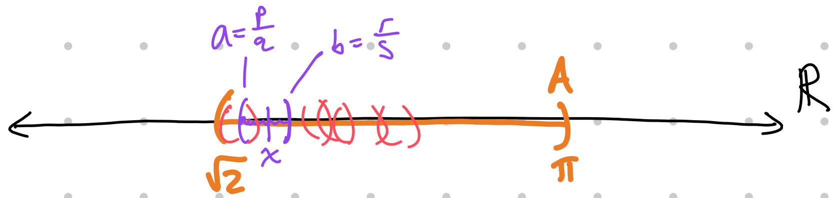
Definition: In a topological space T, a collection $\mathcal{B}$ of open subsets of T is said to form a **basis** for the topology on T if every open subset of T can be written as a union of sets in $\mathcal{B}$.

<u>Example</u>: For the standard topology on $\mathbb{R}$,

$$\mathcal{B} = \{(a,b) \subset \mathbb{R} \mid a,b \in \mathbb{R}\} \quad \text{is a basis.}$$



<u>Example</u>: $\mathcal{B} = \{(a,b) \subset \mathbb{R} \mid a,b \in \mathbb{Q}\}$



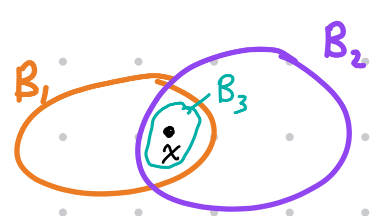
Set A is a union of infinitely many intervals (a,b) with $a,b \in \mathbb{Q}$

Proposition: Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if:

(a) Every point in X is contained in some element of $\mathcal{B}$.

(b) If B_1 and B_2 are sets in $\mathcal{B}$ and x is a point in $B_1 \cap B_2$, then there is a set B_3 in $\mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$

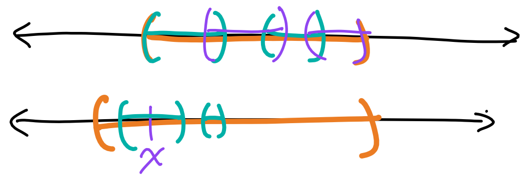


Bases

MATH 348

1. Let $\mathcal{B} = \{(a, b) \cup (c, d) \subset \mathbb{R} \mid a < b < c < d$ are distinct irrational numbers$\}$. Is $\mathcal{B}$ a basis for the standard topology on $\mathbb{R}$?

Yes!



2. Let $\mathcal{B} = \{(n, n + 1) \mid n \in \mathbb{Z}\}$. Is $\mathcal{B}$ a basis for some topology on $\mathbb{R}$?

No.

3. The collection $\mathcal{B} = \{\{x\} \mid x \in \mathbb{R}\}$ is a basis for what topology on $\mathbb{R}$?

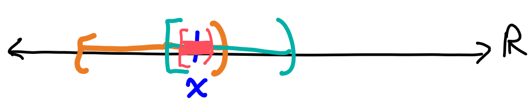
Discrete Topology

4. Let $\mathcal{B} = \{[a, b) \subset \mathbb{R} \mid a < b\}$. Is this a basis for a topology on $\mathbb{R}$? Is this topology the standard topology?

YES. No.

Criteria: (a) Let $x \in \mathbb{R}$. Then $x \in [x-1, x+1)$.

(b)



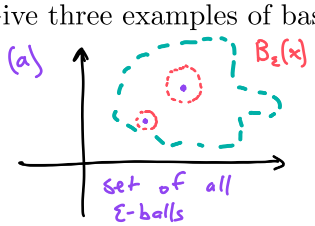
Lower Limit Topology on $\mathbb{R}$, denoted $\mathbb{R}_\ell$

NOTE: $[0,1)$ are open

$(0,1)$ also open

5. Give three examples of bases for the standard topology on $\mathbb{R}^2$

(a)



set of all ε-balls

(b) set of all ε-balls with rational center point and radius

(c) open rectangles

(d) open hearts

6. For each $n \in \mathbb{Z}$ define the set

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd,} \\ \{n-1, n, n+1\} & \text{if } n \text{ is even.} \end{cases}$$

Is $\mathcal{B} = \{B(n) \mid n \in \mathbb{Z}\}$ the basis for a topology on $\mathbb{Z}$? Yes!

? Give an example of a continuous function $f : \mathbb{Z} \to \mathbb{Z}$ in this topology. Then give an example of a discontinuous function.

digital line topology on $\mathbb{Z}$