Connectivity

MATH 348

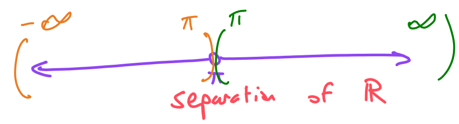
1. Intuitively, which of the following spaces would you say is *connected*?

(a) $\mathbb{R}$ with the standard topology

Yes

(b) $\mathbb{R} - \{\pi\}$ with the subspace topology

No.



(c) S^0 with the subspace topology

No. $\quad S^0 = \{-1, 1\} = \{-1\} \cup \{1\}$

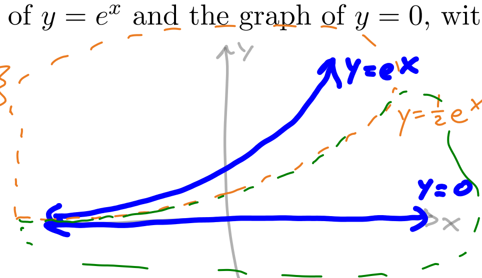
(d) The set $A \subset \mathbb{R}^2$ defined to be the union of the graph of $y = e^x$ and the graph of $y = 0$, with the subspace topology.

No.

Here is a separation:

$U = \{(x,y) \subset \mathbb{R}^2 \mid y > \frac{1}{2}e^x\}$

$V = \{(x,y) \subset \mathbb{R}^2 \mid y < \frac{1}{2}e^x\}$



2. For each space X in #1 that is disconnected, find a separation of X.

3. Is $\mathbb{R}$ with the discrete topology connected?

No. Consider $U = (-\infty, 0]$ and $V = (0, \infty)$.

In the discrete topology, U and V are open.

So $\mathbb{R} = U \cup V$, which is a separation of $\mathbb{R}$.

4. Is $\mathbb{Z}$ with the indiscrete topology connected?

$\hookrightarrow$ only open sets are $\emptyset$ and $\mathbb{Z}$

There is no separation of $\mathbb{Z}$ in this topology,

so $\mathbb{Z}$ is connected.

5. Prove that topological space X is connected if and only if there are no nonempty proper subsets of X that are both open and closed in X.

Equivalently: X is disconnected if and only if there exist nonempty proper subsets of X that are both open and closed.

$\Longrightarrow$ If X is disconnected, then $X = U \cup V$, where U and V are disjoint nonempty open sets. Since X is a disjoint union of U and V, these are complements of each other, so they are closed.

$\Longleftarrow$ If X has a nonempty proper subset U that is both open and closed, let V be the complement of U in X.

So V is open, nonempty (since its complement is proper) and disjoint from U.

Thus, U and V are a separation of X, so X is disconnected.

Definition: We say that a topological space T is **disconnected** if it is possible to find two open subsets U, V of T such that:

– U and V have no intersection, i.e., $U \cap V = \emptyset$ — U and V are disjoint

– U and V cover T, i.e., $U \cup V = T$

– Neither U nor V are empty, i.e., $U \neq \emptyset$ and $V \neq \emptyset$.

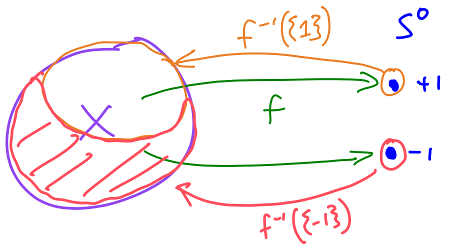
If there are no such subsets within T, then T is **connected**.

Space T is disconnected if it is contained in the union of disjoint, nonempty, open sets.

Such sets are said to be a separation of T.

onto
"fills up the codomain"

6. Let X be a connected topological space. Show that there is no continuous surjective map from X to S^0.



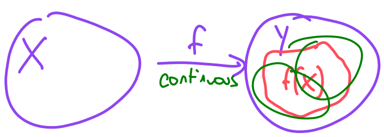
Suppose such a map f exists.
Then $f^{-1}(\{1\})$ and $f^{-1}(\{-1\})$ are disjoint, nonempty, open sets in X whose union is X.
This gives a separation of X, which contradicts the assumption that X is connected.

Therefore, no such map f exists.

7. Let X be a disconnected topological space. Is there a continuous surjective map from X to S^0?

Lemma 4.10

8. Prove that if X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected in Y.



Suppose that $f(X)$ is not connected in Y.
Then there exist U and V that form a separation of $f(X)$.
That is, U and V are nonempty, disjoint open sets that cover $f(X)$.
Consider $f^{-1}(U)$ and $f^{-1}(V)$: these are open (since f continuous), nonempty, disjoint, and cover X.
Thus, $f^{-1}(U)$ and $f^{-1}(V)$ separate X, contradicting the assumption that X is connected.
Therefore, $f(X)$ must be connected.

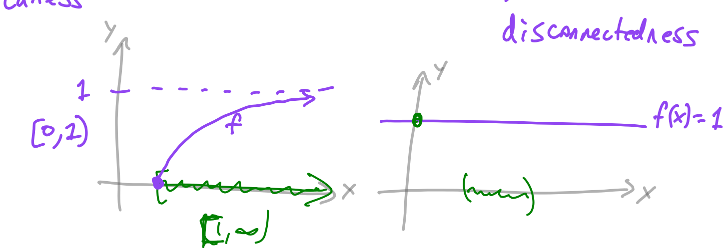
9. What properties of sets are preserved by continuous functions? What properties are not preserved?

Connectedness

openness, closedness
disconnectedness

nonemptyness

closed and bounded sets in $\mathbb{R}$



10. Is there a continuous surjective function from $[0, 1]$ to $(0, 1)$? Why or why not?