26 Sept. 2024

How do you learn mathematics?

higher-order activities

explaining, teaching

doing, solving problems

reading, listening, watching

also necessary:

time, repetition, and learning from mistakes

Connectivity

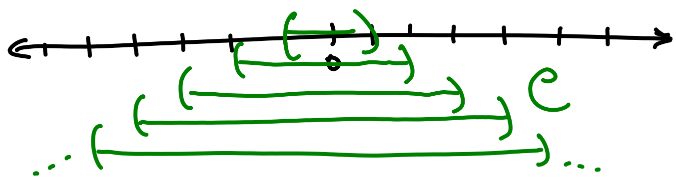
MATH 348

1. Let $\mathcal{C} = \{(-n, n) \mid n \in \mathbb{Z}\}$.

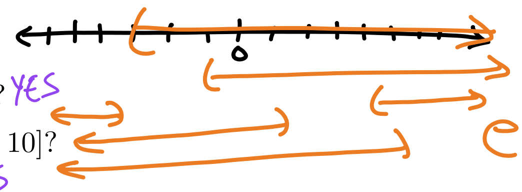
(a) Does $\mathcal{C}$ cover $\mathbb{R}$? YES — Is every $x \in \mathbb{R}$ in one of the sets in $\mathcal{C}$?

(b) Does any finite subcollection of $\mathcal{C}$ cover $\mathbb{R}$? NO

(c) Does any finite subcollection of $\mathcal{C}$ cover $[0, 10]$? YES. consider $(-11, 11)$



2. Let $\mathcal{C} = \{(n, \infty) \mid n \in \mathbb{Z}\} \cup \{(-\infty, n) \mid n \in \mathbb{Z}\}$.



(a) Does $\mathcal{C}$ cover $\mathbb{R}$? YES

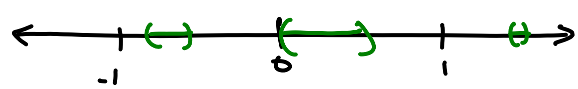
(b) Does any finite subcollection of $\mathcal{C}$ cover $\mathbb{R}$? YES

(c) Does any finite subcollection of $\mathcal{C}$ cover $[0, 10]$? YES

length $2^{-n} = \frac{1}{2^n}$ — positive integers $\mathbb{N}$

possible lengths $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

3. Let $\mathcal{C} = \{(x, x + 2^{-n}) \mid x \in \mathbb{R}, n \in \mathbb{Z}_+\}$.



(a) Does $\mathcal{C}$ cover $\mathbb{R}$? Yes.

(b) Does any finite subcollection of $\mathcal{C}$ cover $\mathbb{R}$? No.

(c) Does any finite subcollection of $\mathcal{C}$ cover $[0, 10]$? Yes.

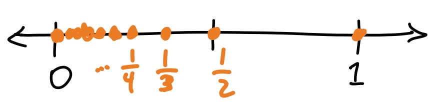
Finite refinement of the cover.
"subcover"

4. Prove that any finite topological space is compact.

$X = \{a_1, a_2, a_3, \ldots, a_N\}$ finite list of elements

Suppose $\mathcal{C}$ is an open cover of X.
Then every element of X is in some open set in $\mathcal{C}$.
Let A_x be an open set in $\mathcal{C}$ that contains x. So $\{A_{a_1}, A_{a_2}, \ldots, A_{a_N}\}$ is a finite cover of X.

5. Let $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$. Is A compact in $\mathbb{R}$?



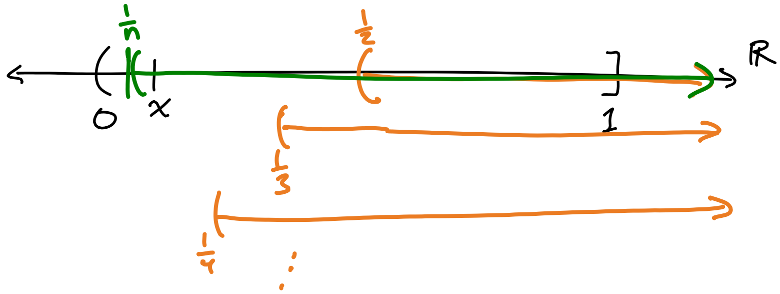
Note: $\{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ is not compact

Suppose $\mathcal{C}$ is an open cover of A.
Then some element $S \in \mathcal{C}$ contains 0.
S is an open set containing 0, so it contains an open interval around 0. This open interval contains all but finitely many points in A.
Thus, $\mathcal{C}$ has a finite refinement.

Definition: An **open cover** of a topological space T is a collection of open subsets of T such that every point in T lies in at least one of these open subsets.

A topological space T is said to be **compact** if every open covering of T admits a finite refinement. In other words, given any infinite collection of open sets, which covers T, it is possible to throw most of these sets away, keeping only a finite number of them, and still have an open covering of T.

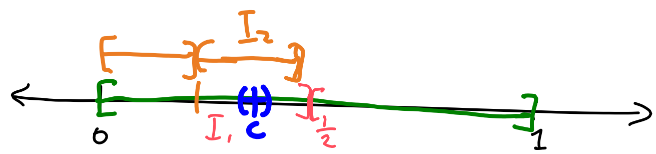
6. Is $(0, 1]$ compact as a subspace of $\mathbb{R}$? No.



$\mathcal{C} = \{(\frac{1}{n}, \infty) \mid n \in \mathbb{Z}_+\}$

↑ $\mathcal{C}$ is an open cover with no finite refinement

Yes! **7.** Is $[0, 1]$ compact as a subspace of $\mathbb{R}$?



Suppose open cover $\mathcal{C}$ has no finite refinement.

Consider $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. One of these must not have a finite refinement. Call this interval I_1.

Divide I_1 in half, then one of its subintervals has no finite refinement, call it I_2

Continuing in this way, we obtain a nested sequence

$$[0,1] \supset I_1 \supset I_2 \supset I_3 \supset \cdots$$

where I_n has length $\frac{1}{2^n}$ and no finite refinement of the open cover.

Now $\bigcap_{i=1}^{\infty} I_i$ is a limit point $c \in [0,1]$, which is in some open set of the cover. So $c \in (c-\delta, c+\delta)$ for some $\delta > 0$, which is contained in an open set of the cover. But I_n is contained in $(c-\delta, c+\delta)$ for sufficiently large n. This contradicts the statement that I_n has no finite refinement. Thus, $\mathcal{C}$ has a finite refinement, and $[0,1]$ is compact.

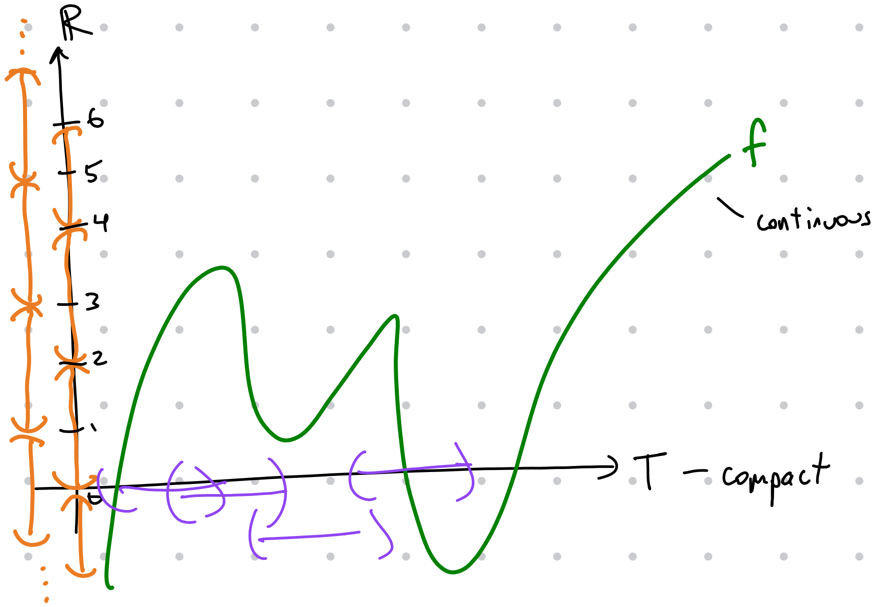
Important theorems:

- Prop. 4.22 If T is a compact topological space and $f : T \to \mathbb{R}$ is a continuous function, then f is bounded.
- Prop. 4.27 If $f : S \to T$ is a continuous map and S is compact, then the image of f is compact.
- A subspace T of $\mathbb{R}^n$ is compact if and only if T is closed (as a subset of $\mathbb{R}^n$) and bounded.

Heine-Borel Theorem (Thm 4.29)

Challenge Problem: Let the set of rationals $\mathbb{Q}$ have the subspace topology from $\mathbb{R}$. Find a set $S \subset \mathbb{Q}$ that is closed and bounded but not compact.

Sketch of Proof of Prop. 4.22



Cover:

$\mathcal{C} = \{(n, n+2) \mid n \in \mathbb{Z}\}$

Consider preimage $f^{-1}(A)$ for each $A \in \mathcal{C}$

This gives a cover of T, which has a finite refinement.

This tells us that the image of f is contained in finitely many elements of $\mathcal{C}$.