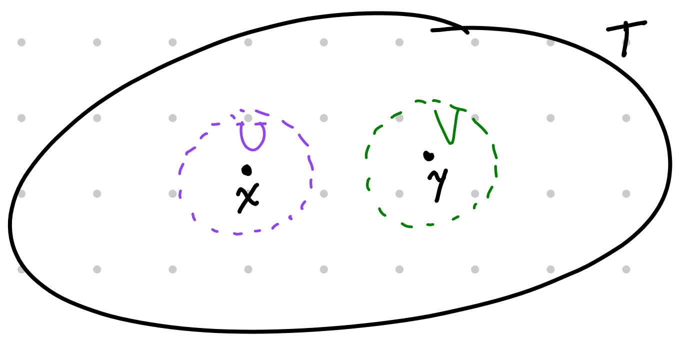
1 October 2024

Section 4.3 of the text:

Definition: We say that a topological space T is **Hausdorff** if, for any two distinct points x, y in T, there are open subsets U, V of T such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.



Distinct points can be "housed off" from each other.

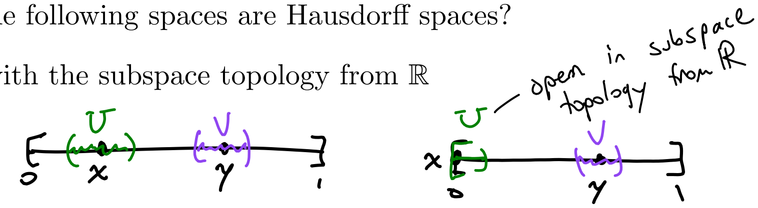
The Hausdorff Property

MATH 348

1. Which of the following spaces are Hausdorff spaces?

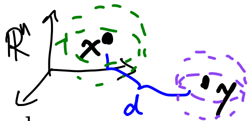
(a) $[0, 1]$ with the subspace topology from $\mathbb{R}$

Yes



(b) $\mathbb{R}^n$ with the standard topology

Yes



Consider open balls of radius $\leq \frac{d}{2}$ around x and y.

$U = B_{d/3}(x)$ $\qquad V = B_{\frac{d}{3}}(y)$

(c) $\{a, b\}$ with the indiscrete topology

No The only nonempty open set is $\{a,b\}$.

(d) Any set X with the discrete topology

↳ every set is open

Yes

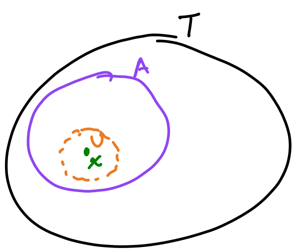
If $x, y \in X$, then let $U = \{x\}$ and $V = \{y\}$

NOTE: point x is distinct from set $\{x\}$

(e) Set $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, X\}$

No For example: If $x = a$ and $y = b$, then there is no open set U containing a but not b.

2. Let T be a topological space. Suppose A is a subset of ~~S~~ T such that for every $x \in A$ there exists open set U such that $x \in U \subset A$. Prove A is open.



proof: For every $x \in A$, there exists open set U_x such that $x \in U_x \subset A$.

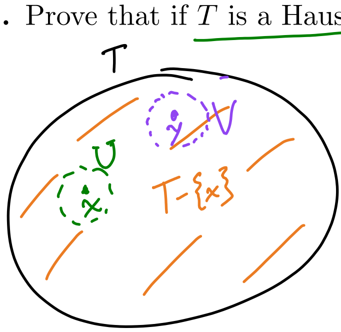
Consider $\bigcup_{x \in A} U_x$. Claim: this union equals A.

Since $U_x \subset A$ for all x, $\boxed{\bigcup_{x\in A} U_x \subset A}$.

For any $x \in A$, $x \in U_x \subset \bigcup_{x \in A} U_x$, thus $\boxed{A \subset \bigcup_{x \in A} U_x}$

Thus, $\bigcup_{x \in A} U_x = A$. Thus, A is a union of open sets, and thus open.

3. Prove that if T is a Hausdorff space, then every single-point subset of T is closed.



$\{x\}$

We must show that $T - \{x\}$ is open.

Let $y \in T - \{x\}$. So x and y are distinct.

Since T is Hausdorff, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$.

Now $x \notin V$, since $x \in U$ and $U \cap V = \varnothing$.

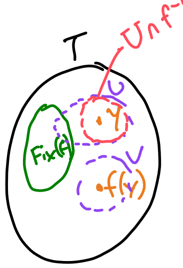
Thus, $V \subset T - \{x\}$.

So we have every $y \in T - \{x\}$ has a open set V such that $y \in V \subset T - \{x\}$, and by the previous problem, $T - \{x\}$ is open.

Therefore, $\{x\}$ is closed in T.

4. If T is a topological space and $f : T \to T$ is a continuous map, then the **fixed-point set** of f is defined $\mathrm{Fix}(f) = \{x \in T \mid f(x) = x\}$.

Prove that if T is a Hausdorff space then $\mathrm{Fix}(f)$ is a closed subset of T.



We will show that $T - \mathrm{Fix}(f)$ is open.

Let $y \in T - \mathrm{Fix}(f)$. So $y \neq f(y)$. So y and $f(y)$ are distinct points in T.

Since T is Hausdorff, there exist open sets U and V such that $y \in U$, $f(y) \in V$, and $U \cap V \neq 0$. Since f is continuous, $f^{-1}(V)$ is open.

Consider $U \cap f^{-1}(V)$. Note $y \in U \cap f^{-1}(V)$.

Suppose $x \in U \cap f^{-1}(V)$ and $x \in \mathrm{Fix}(f)$. Since $x \in f^{-1}(V)$, $f(x) \in V$.
But also, $x \in U$ and $f(x) = x$, so $f(x) \in U$. Thus, $f(x) \in U \cap V$, which contradicts $U \cap V = \emptyset$.
Therefore, there is no x in $U \cap f^{-1}(V)$ and in $\mathrm{Fix}(f)$.

Thus, for every $y \in T - \mathrm{Fix}(f)$, there is an open set containing y and contained in $T - \mathrm{Fix}(f)$, $T - \mathrm{Fix}(f)$ is open, and so $\mathrm{Fix}(f)$ is closed.

5. For integers $a \neq 0$ and b, let $S(a, b) = \{an + b \mid n \in \mathbb{Z}\}$.

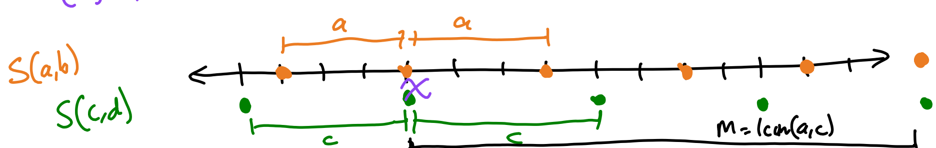
(a) Show that the collection $\{S(a, b) \mid a, b \in \mathbb{Z}, a \neq 0\}$ is the basis for a topology on $\mathbb{Z}$. This topology is called the **arithmetic sequence topology**.

Criteria:

(i) Every integer is in a basis element: $S(1,0) = \mathbb{Z}$

(ii) An intersection of two basis elements is a basis element: Let $x \in S(a,b) \cap S(c,d)$

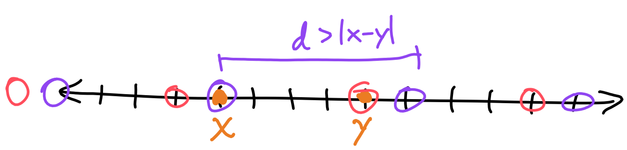
Let $m = \mathrm{lcm}(a,c)$.



Then $S(m,x) \subset S(a,b) \cap S(c,d)$

(b) Show that the arithmetic sequence topology is Hausdorff.

Let x, y be distinct integers.



Let d be an integer greater than $|x-y|$. Then $U = S(d,x)$ and $V = S(d,y)$ are open sets that show Hausdorff property.

(c) Show that the basis elements are both open and closed in this topology.

Basis elements are open by definition.

$S(a,b)$ is closed because its complement is open:

$$S(a,b) = \mathbb{Z} - \bigcup_{j=1}^{a-1} S(a, b+j)$$

(d) Use topology to prove the infinitude of primes as follows:

Let $Q = \bigcup_{p \text{ prime}} S(p, 0)$. Assume that there are only finitely many primes, and explain why this implies that Q is closed.

Then $\mathbb{Z} - Q$ must be open. What integers are elements of $\mathbb{Z} - Q$? Why is this a contradiction?

Since each $S(p,0)$ is closed, Q is a finite union of closed sets and thus is closed. (Its complement is a finite intersection of open sets.)

So $\mathbb{Z} - Q$ must be open.

Now $\mathbb{Z} - Q = \{-1, +1\}$. But $\{-1, 1\}$ contains no basis elements, so it cannot be open

The only integers that are not multiples of any prime.

Arithmetic sequence:

common difference

examples: $\{\ldots, 2, 4, 6, 8, 10, 12, \ldots\} = \{2n+0 \mid n \in \mathbb{Z}\}$

$\{\ldots, 6, 11, 16, 21, 26, \ldots\} = \{5n+1 \mid n \in \mathbb{Z}\}$

$\{\ldots, 3, 7, 11, 15, 19, \ldots\} = \{4n+3 \mid n \in \mathbb{Z}\}$

If $a \neq 0$, and $a, b \in \mathbb{Z}$, then: $\{an+b \mid n \in \mathbb{Z}\}$ contains the elements of an arithmetic sequence.