8 October 2024

topologically equivalent

> Definition: Two topological spaces S and T are said to be **homeomorphic** if there are continuous maps $f : S \to T$ and $g : T \to S$ such that
>
> $$(f \circ g) = id_T \qquad \text{and} \qquad (g \circ f) = id_S.$$
>
> identity on T   identity on S
>
> The maps f and g are then **homeomorphisms**. These maps are inverse to each other, so we may write f^{-1} in place of g and g^{-1} in place of f. If S and T are homeomorphic, then we write $S \cong T$.

\cong "congruent"

$$\begin{tikzcd} S \arrow[r, bend left, "f"] & T \arrow[l, bend left, "g"] \end{tikzcd}$$

id_T: for any $x \in T$, $id_T(x) = x$

An identity map takes any point to itself.

f and g are inverses

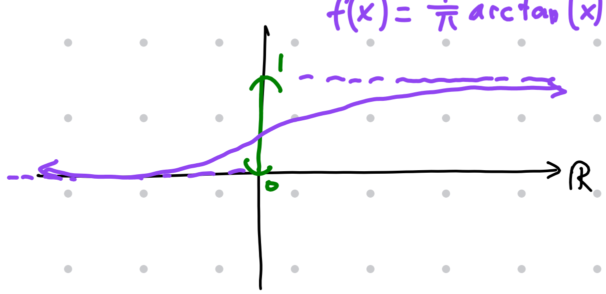
$f = g^{-1}$ and $g = f^{-1}$

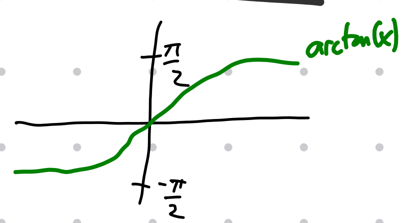
In other words, S and T are homeomorphic if there exists a continuous bijection $f: S \to T$ with a continuous inverse.

Moreover, S and T are homeomorphic if you can continuously map both points and open sets from one to the other.

$(0,1) \cong \mathbb{R}$

$f(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$





Homeomorphisms

MATH 348

1. Which of the following spaces do you think we should regard as topologically *the same*? What should "topologically the same" mean?

(a) $(0,1)$

(b) $[0,1]$

(c) $(0,1]$

(d) S^1 ?

(e) $\mathbb{R}$

(f) $[0,2\pi)$

(g) $\mathbb{R}\cup\{\infty\}$

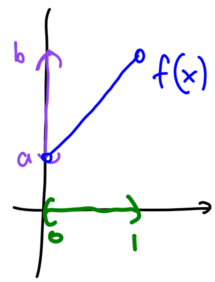
What is the topology?

(h) (a,b) for any $a<b\in\mathbb{R}$

(i) $[a,b]$ for any $a<b\in\mathbb{R}$

EXAMPLES OF HOMEOMORPHISMS

- $f\ (0,1) \to (a,b)$ continuous, bijective, continuous inverse

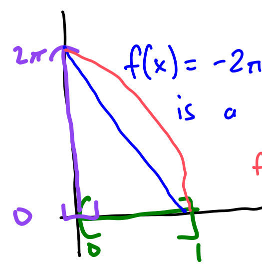


$f(x) = \frac{b-a}{1}x + a$

$(0,1) \cong (a,b)$

- similarly, $[0,1] \cong [a,b]$

- $(0,1] \cong [0,2\pi)$



$f(x) = -2\pi x + 2\pi$ is a homeomorphism

$f(x) = 2\pi(1-x^2)$

2. Find three different topologies on the three-point set $\{a,b,c\}$, each consisting of five open sets, such that two of the topologies are homeomorphic, but the third is not homeomorphic to the other two.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & T \arrow[r, "g"] & U \end{tikzcd}$$

3. Let $f : S \to T$ be a homeomorphism and $g : T \to U$ be another homeomorphism. Show that $g \circ f : S \to U$ is a homeomorphism.

↳ Need to show that $g\circ f$ is continuous with a continuous inverse.

First, composition of continuous maps is continuous. (Prop. 3.10)

Second, since f and g are homeomorphisms, $f^{-1} : T \to S$ and $g^{-1} : U \to T$ are continuous.

So $f^{-1} \circ g^{-1}$, the inverse of $g\circ f$, is also continuous.

Third, the composition of bijections is bijective, so $g\circ f$ is bijective.

4. Let S and T be homeomorphic spaces. Prove each of the following:

(a) If S is connected, then so is T.

(b) If S is compact, then so is T.

(c) If S is Hausdorff, then so is T.

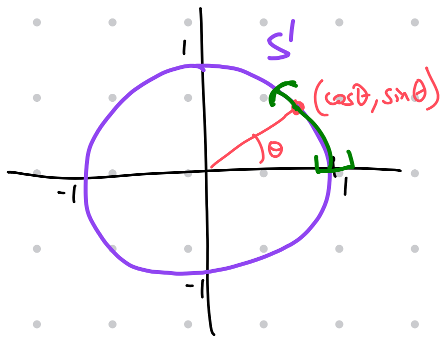
This is Proposition 5.16 in the text.

5. Suppose $f : X \to Y$ is a continuous bijective function. Is f^{-1} necessarily continuous?

No. See below.

Is S^1 homeomorphic to $[0, 2\pi)$?

subspace topology from standard topology on $\mathbb{R}$



$\theta \in [0, 2\pi)$

$f: [0, 2\pi) \to S^1$

$f(\theta) = (\cos(\theta), \sin(\theta))$ is a bijection

is continuous

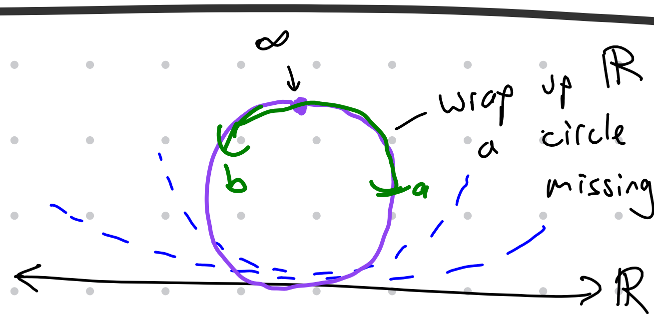
Is its inverse continuous? No.

consider $f([0, \frac{1}{2}))$ = , which is not open in S^1.

open in subspace topology

$\mathbb{R} \cup \{\infty\}$

"point at infinity"



With a certain topology, $\mathbb{R} \cup \{\infty\} \cong S^1$

basis : open intervals in $\mathbb{R}$

and sets of the form $(a, \infty) \cup \{\infty\} \cup (-\infty, b)$

See Example 5.7 in the text.