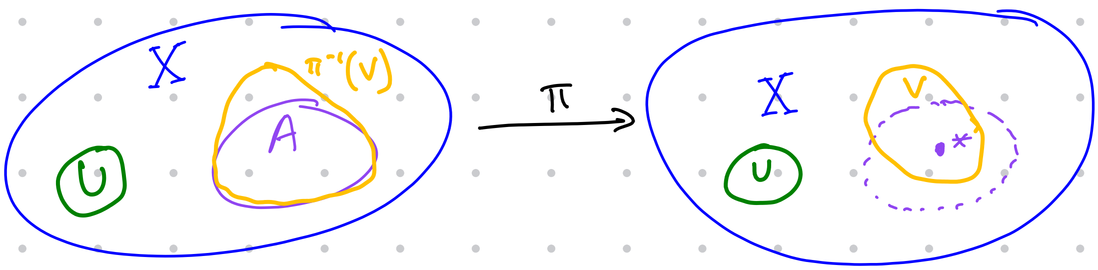
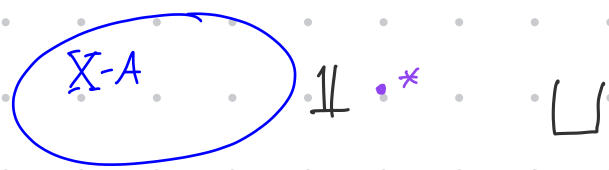
22 October 2024

If X is a topological space and A is a nonempty subspace of X, then the quotient space is the set $(X-A) \amalg \{*\}$. A subset $U \subset (X-A) \amalg \{*\}$ is open if and only if $\pi^{-1}(U)$ is open in X, where $\pi : X \to (X-A) \amalg \{*\}$ is defined by $\pi(x) = x$ if $x \notin A$ and $\pi(x) = *$ if $x \in A$.





Equivalence Relation on a Set:

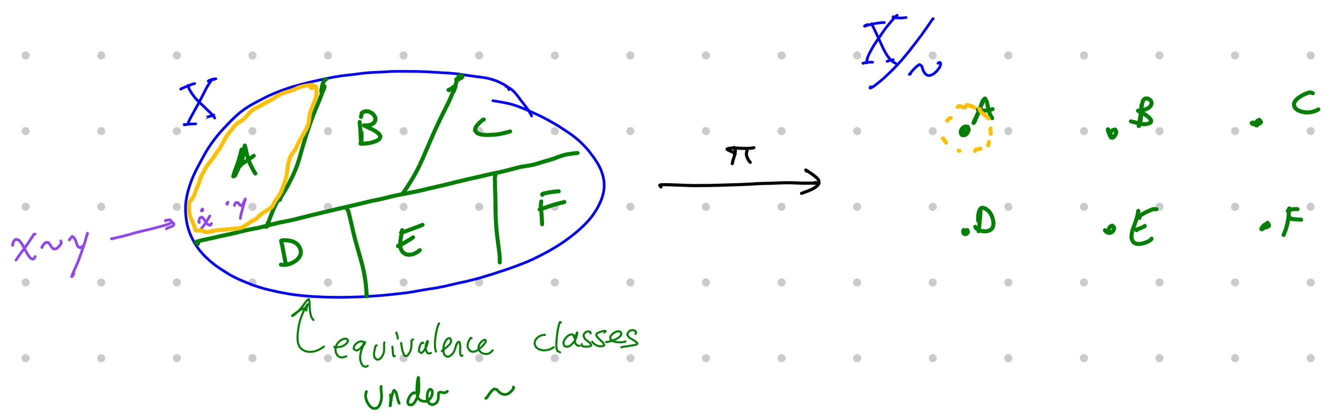
Set X. A relation is a set of pairs of elements of X, with 3 properties

↳ $x \sim y$

`\sim`

- Reflexive: $x \sim x$
- Symmetric: $x \sim y$ iff $y \sim x$
- Transitive: if $x \sim y$ and $y \sim z$, then $x \sim z$

If $\sim$ is an equivalence relation on a set X, then we can form a quotient space $X/\sim$ with a quotient function $\pi : X \to X/\sim$ that takes an element x to its equivalence class.
A subset $U \subset X/\sim$ is open if and only if $\pi^{-1}(U)$ is open.



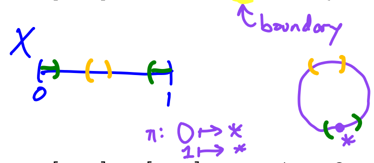
Quotient Spaces

MATH 348

1. Describe each quotient space given by X/A or $X/\sim$.

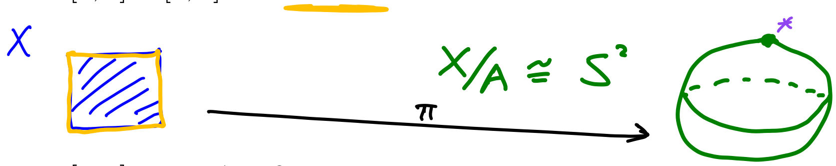
(a) $X = [0,1]$ and $A = \partial X = \{0,1\}$

boundary



$X/A \cong S^1$

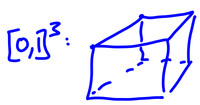
(b) $X = [0,1] \times [0,1]$ and $A = \partial X$



$X/A \cong S^2$

(c) $X = [0,1]^n$ and $A = \partial X$

$[0,1]^3$:

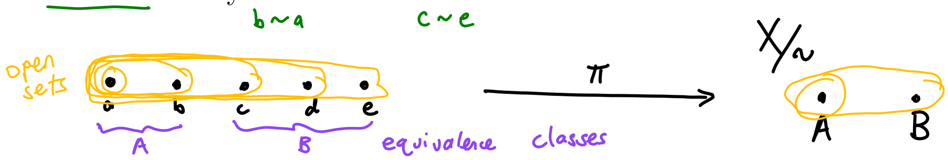


$X/A \cong S^n$ (n-dimensional sphere)

(d) $X = \{a,b,c,d,e\}$ with the topology $\{\varnothing, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,c,d\}, X\}$. Define an equivalence relation on X by $a \sim b$ and $c \sim d \sim e$.

$a \sim a$, $b \sim b$, ...

$b \sim a$ $\quad$ $c \sim e$



equivalence classes

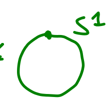
think. $A = \{a,b\}$ $\quad$ $B = \{c,d,e\}$

(e) $X = [0,1]$ with $x \sim x'$ if $x - x' \in \mathbb{Z}$

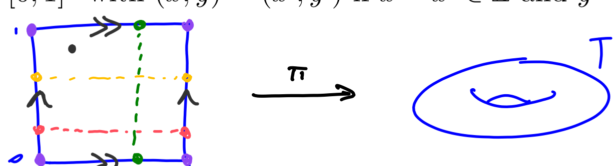
$0 \sim 1$ since $0 - 1 \in \mathbb{Z}$

If $x \in [0,1]$, then $x \sim x$ since $x - x = 0 \in \mathbb{Z}$

$X/\sim \; \cong$



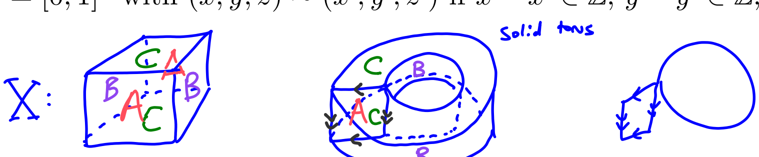
(f) $X = [0,1]^2$ with $(x,y) \sim (x',y')$ if $x - x' \in \mathbb{Z}$ and $y - y' \in \mathbb{Z}$



$T^2 \approx S^1 \times S^1$

(g) $X = [0,1]^3$ with $(x,y,z) \sim (x',y',z')$ if $x - x' \in \mathbb{Z}$, $y - y' \in \mathbb{Z}$, and $z - z' \in \mathbb{Z}$

Solid torus

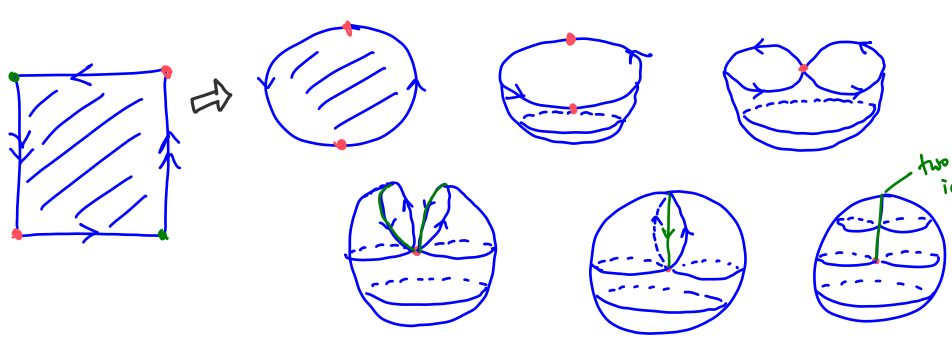


$S^1 \times T^2$

$\cong S^1 \times S^1 \times S^1 \cong T^3$

3-torus

(h) $X = [0,1]^2$ with each side identified with the opposite side with directions reversed



two separate identifications of edges

real projective plane

$\mathbb{RP}^2$

to be continued ..

2. Suppose X is a topological space with equivalence relation $\sim$. Let Q be the quotient space $X/\sim$, and let $\pi : X \to Q$ be the projection (quotient) map.

(a) If S is any other topological space and $f : Q \to S$ is continuous, then there exists a map $g : X \to S$ such that the following diagram commutes.

(b) If $g : X \to S$ is any continuous function such that $g(x) = g(y)$ whenever $x \sim y$, then there is a continuous function $f : Q \to S$ such that the following diagram commutes.