24 October 2024

Worksheet continued from last time:

2. Suppose X is a topological space with equivalence relation $\sim$. Let Q be the quotient space $X/\sim$, and let $\pi : X \to Q$ be the projection (quotient) map.

(a) If S is any other topological space and $f : Q \to S$ is continuous, then there exists a continuous map $g : X \to S$ such that the following diagram commutes.

$\hookrightarrow$ $g(x) = f(\pi(x))$ for all $x \in X$

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ Q = X/\sim \arrow[r, "f"'] & S \end{tikzcd}$$

Proof: Define $g(x) = f(\pi(x))$.

Since f and π are continuous, g is also continuous as a composition of continuous maps (Proposition 3.10).

(b) If $g : X \to S$ is any continuous function such that $g(x) = g(y)$ whenever $x \sim y$, then there is a continuous function $f : Q \to S$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ Q = X/\sim \arrow[r, dashed, "f"'] & S \end{tikzcd}$$

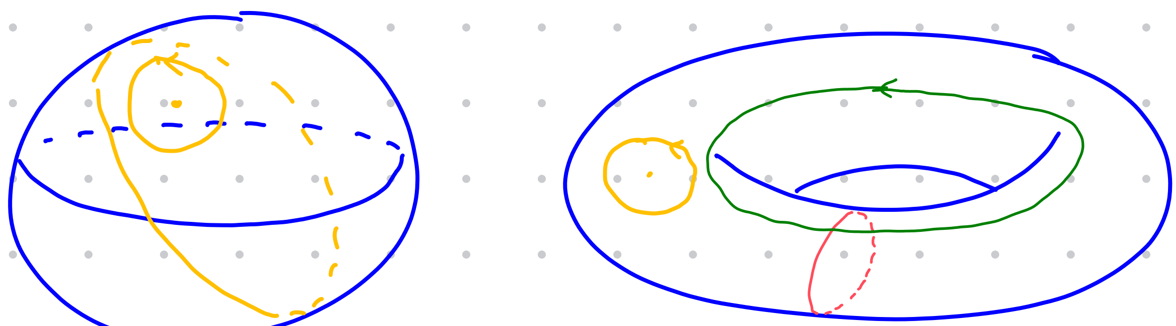
Define $f : Q \to S$ by $f(E) = g(x)$ if E is the equivalence class containing x.
Note that if $y \in E$, then $x \sim y$, so $g(x) = g(y)$, so $f(E) = g(y)$. Thus, it does not matter which element x of E we choose, and f is well-defined.
So $g = f \circ \pi$.

To show f is continuous, let R be an open set in S
Then: $\quad g^{-1}(R) = (f \circ \pi)^{-1}(R) = \pi^{-1}(f^{-1}(R))$

Since g is continuous, $g^{-1}(R)$ is open. That is, $\pi^{-1}(f^{-1}(R))$ is open.

The topology on Q is defined so that π^{-1} of a set is open $\underbrace{\text{precisely if}}_{\text{if and only if}}$ the set is open, so $\pi^{-1}(f^{-1}(R))$ bein open means that $f^{-1}(R)$ is open. Thus, f is continuous.

Motivation for Homotopy



S^2 $\qquad\qquad$ T^2

How to distinguish between these?

F interpolates between f and g.

<u>Definition:</u> Two maps $f, g : S \to T$ are homotopic if there is a continuous function $F : S \times [0,1] \to T$ such that $F(s, 0) = f(s)$ for all $s \in S$ and $F(s, 1) = g(s)$ for all $s \in S$.

In this case, F is a homotopy between f and g, and we write $f \simeq g$.

symbol $\simeq$ is \simeq