31 October 2024

Question: What does it mean for two spaces to be homeomorphic? ... homotopy equivalent?

Homeomorphic: continuous bijection with a continuous inverse

$\Downarrow$

Homotopy invariant: There exist maps $f: S \to T$ and $g: T \to S$ such that $g \circ f$ is homotopic to id_S and $f \circ g$ is homotopic to id_T

4. Explain which of the following pairs of spaces are homotopy equivalent.

Contractible spaces

(a) The single point space $\{0\}$ and $\mathbb{R}$

Note: $\{0\} \simeq \mathbb{R}$ homotopy equivalent

$\{0\} \not\cong \mathbb{R}$ NOT homeomorphic

(b) The closed interval $[0, 1]$ and the single point space $\{0\}$

$[0,1] \simeq \{0\}$

$f: [0,1] \to \{0\}$ defined $f(x) = 0$

$g: \{0\} \to [0,1]$ define $g(0) = 0$

Compositions

$f \circ g: \{0\} \to \{0\}$ is $id_{\{0\}}$

$g \circ f: [0,1] \to [0,1]$ sends $x \in [0,1]$ to $0 \in [0,1]$

the constant map

$g \circ f \simeq id_{[0,1]}$

Homotopy: $F(x,t) = x \cdot t$

(c) The open interval $(0, 1)$ and the single point space $\{0\}$

$(0,1) \simeq \{0\}$

$f: (0,1) \to \{0\}$ defined $f(x) = 0$

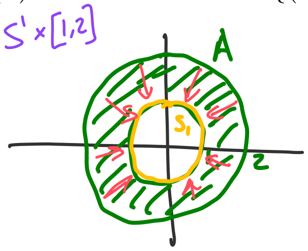
$g: \{0\} \to (0,1)$ defined $g(0) = 0.74$

$f \circ g = id_{\{0\}}$

$g \circ f \simeq id_{[0,1]}$

(d) The annulus $A = \{(x, y) \in \mathbb{R} \mid 1 \leq \sqrt{x^2 + y^2} \leq 2\}$ and the circle S^1

$A = S^1 \times [1,2]$



$f: S^1 \to A$ defined $f(x,y) = (x,y)$ is the identity map or inclusion map

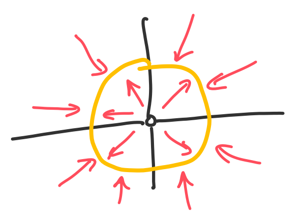
$g: A \to S^1$ defined $g(x,y) = \frac{1}{\sqrt{x^2+y^2}}(x,y)$

$g \circ f = id_{S_1}$ and $f \circ g \simeq id_A$

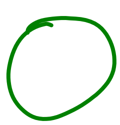
$A \simeq S^1$

(e) The punctured plane $\mathbb{R}^2 - \{(0, 0)\}$ and the circle S^1

Yes: $\mathbb{R}^2 - \{(0,0)\} \simeq S^1$



(f) S^1 and S^0 NOT HOMOTOPY EQUIV.

S^1  $\not\simeq$ $S_0 = \{-1, +1\}$

→ Proposition 6.19 in the text

CLAIM: The 2-point space S^0 is not contractible.
That is, $S^0 = \{-1, +1\}$ is not homotopy equiv. to $\{0\}$.

proof: Suppose S^0 is contractible. That is, there exist maps
$$f: S^0 \to \{0\} \quad \text{and} \quad g: \{0\} \to S^0$$
such that $f \circ g \simeq id_{\{0\}}$ and $g \circ f \simeq id_{S^0}$.
In fact, $f \circ g = id_{\{0\}}$
There is a homotopy from $g \circ f$ and id_{S^0}:
$$F: S^0 \times I \to S^0$$
with $F(x,0) = x$ and $F(x,1) = g(f(x)) = g(0)$.

Define a map $h: [0,1] \to S^0$ by $h(t) = F(-g(0), t)$.
So: $h(0) = F(-g(0), 0) = \underline{-g(0)}$ and $h(1) = F(-g(0), 1) = \underline{g(0)}$.
Thus, h is surjective onto S^0.
Also, h is continuous as a composition of continuous functions
So h is a continuous surjection from $[0,1]$, which is connected,
to S^0, which is disconnected. This contradicts Lemma 4.3.
Therefore, S^0 is not contractible.

Proposition 6.20: If X is connected and Y is disconnected, then X and Y are not homotopy equivalent.