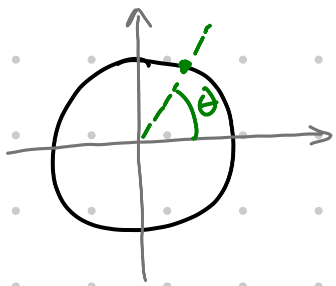
5 November 2024

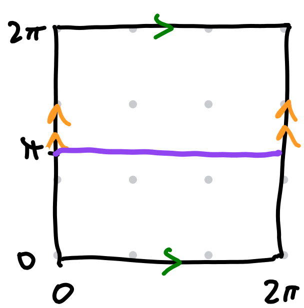
A **circle map** is a function $f: S^1 \to S^1$.

Represent points on the circle by an angular variable θ.



Two values θ_1 and θ_2 represent the same point if they differ by an integer multiple of 2π.

The graph of f is the set $\{(\theta, f(\theta)) \in S^1 \times S^1\}$ which is a subset of the torus $S^1 \times S^1$. Consider the torus as $[0, 2\pi] \times [0, 2\pi]$ with opposite edges identified. Then we can draw circle maps on squares.



Example: $f(\theta) = \pi$

The **degree** (or **winding number**) of a circle map is how many times it wraps around the circle.

In particular:

$f(\theta) = \theta$ has degree 1

$f(\theta) = 2\theta$ has degree 2

$\vdots$

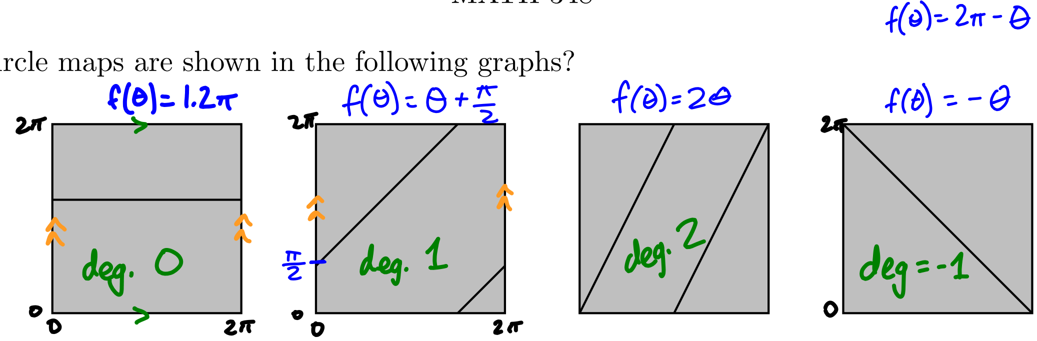
$f(\theta) = n\theta$ has degree n for any $n \in \mathbb{Z}$

$f(\theta) = -\theta$ has degree -1

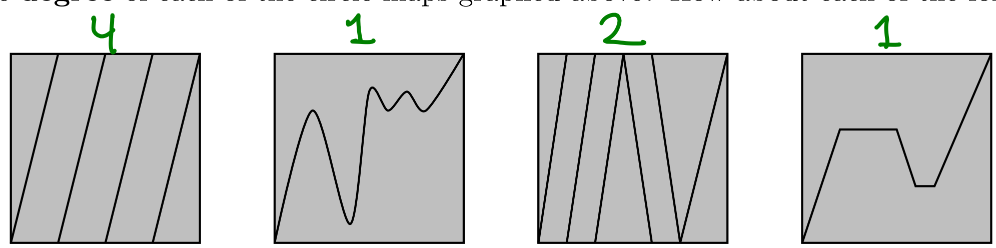
Circle Maps

MATH 348

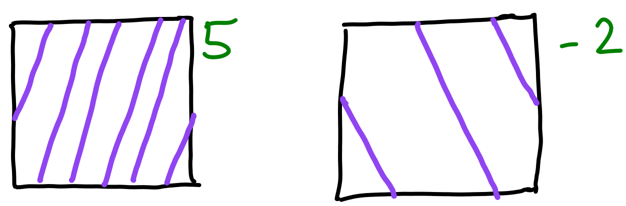
1. What circle maps are shown in the following graphs?



2. What is the **degree** of each of the circle maps graphed above? How about each of the following?



3. Sketch the graph of a circle map with degree 5. Then sketch a circle map with degree -2.



4. Prove the statement: If a circle map $f : S^1 \to S^1$ has a nonzero degree, then it is surjective.

CONTRAPOSITIVE: If f is not surjective, then it misses some point on S^1, so it can't wrap around S^1, meaning it has zero degree.

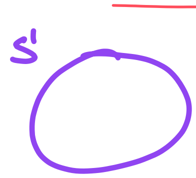
5. Under what condition, do you think, are two circle maps are homotopic? Why?

$f \simeq g$ if and only if f and g have the same degree

Section 6.3 in Crossley text

6. Is the circle S^1 contractible? Can you prove your answer?

No.



Proof by contradiction:

Suppose $f: S^1 \to \{0\}$ and $g: \{0\} \to S^1$ are homotopy equivalences. This means that $(g \circ f)(\theta) = g(0)$ for all $\theta \in S^1$, so $(g \circ f)$ is constant.

So $g \circ f : S^1 \to S^1$ is a map with degree 0.

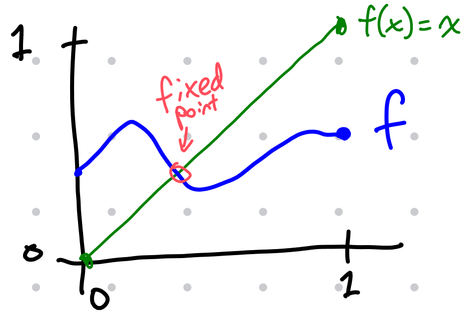
But the identity map on S^1 has degree 1

So $g \circ f$ cannot be homotopic to the identity, so we don't have a homotopy equivalence. Thus, S^1 is not contractible

Brouwer Fixed-Point Theorem: If f is a continuous function from a nonempty compact convex set to itself, then f has a fixed point.

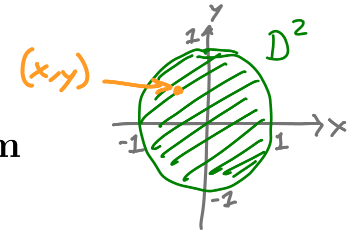
x such that $f(x)=x$.

1-D version: If $f: [0,1] \to [0,1]$ is continuous, then f has a fixed point.



proof: define $g(x) = f(x) - x$

Use intermediate value theorem to show $g(x)=0$ somewhere



2D Brouwer's Fixed-Point Theorem

MATH 348

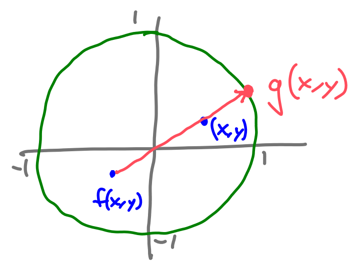
2-D Brouwer's Fixed-Point Theorem: Let $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the closed disk, and let $f : D^2 \to D^2$ be a continuous map. Then there is some point $(x,y) \in D^2$ such that $f(x,y) = (x,y)$.

Complete the following steps to prove Brouwer's fixed-point theorem in 2-D:

Proof by contradiction

(a) Suppose $f : D^2 \to D^2$ does not have a fixed point. Explain why this means that we can draw a line through $f(x,y)$ and (x,y) and extend this line through (x,y) until it meets the boundary of D^2.

Then $f(x,y) \neq (x,y)$ for all $(x,y) \in D^2$.



Note: if (x,y) is on the boundary, then $g(x,y) = (x,y)$

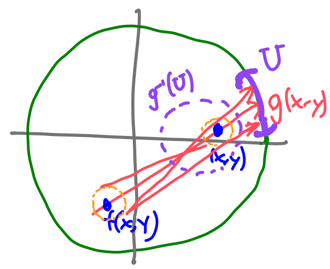
(b) Define a function $g : D^2 \to S^1$ such that $g(x,y)$ is the point where the line from $f(x,y)$ through (x,y) meets the boundary of D^2. Explain why g is continuous.

intuition: A small change in (x,y) produces a small change in $f(x,y)$ since f is continuous. This small change in (x,y) and $f(x,y)$ results in a small change to where the line hits the circle.

Let $U \subset S^1$ be open.

Consider $g^{-1}(U)$.

See page 121 in text for details.



(c) Define a map

$$F : S^1 \times I \to S^1$$

by $F(\overbrace{(x,y)}^{S^1}, \overbrace{t}^{I}) = g(tx, ty)$. Argue that F is a homotopy between maps $h(x,y) = F((x,y), \overset{t=0}{0})$ and $j(x,y) = F((x,y), \underset{t=1}{1})$. Describe the maps h and j.

$h : S^1 \to S^1$ is $h(x,y) = F((x,y), 0) = g(0x, 0y) = g(0,0)$ is a constant map

So deg of h is zero!

$j : S^1 \to S^1$ is $j(\underset{(x,y) \in S^1}{\underline{x,y}}) = F((x,y), 1) = g(1x, 1y) = g(x,y) = (x,y)$ is the identity on S^1

So deg of j is 1.

(d) What are the degrees of the maps h and j? How does this produce a contradiction?

Contradiction: there is no homotopy between h of deg. 0 and j of deg. 1.

Therefore, f has a fixed point.