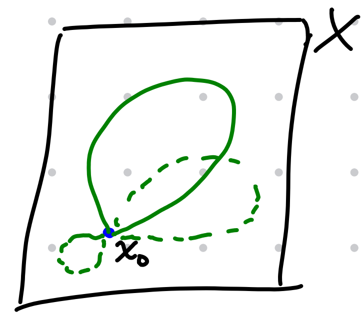
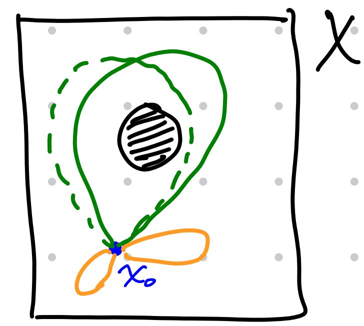
12 November 2024

Let X be a topological space and $x_0 \in X$ a designated base point.

A *loop* based at x_0 is a map $f: [0,1] \to X$ with $f(0) = f(1) = x_0$



The *fundamental group* $\pi_1(X, x_0)$ is defined to be the set of homotopy classes of loops based at x_0.



This is an algebraic group under the operation of path concatenation:

"$f + g$" means follow path f, then follow path g

If X is "path connected", then we don't have to specify a basepoint

EXAMPLES:

- If X is contractible, then $\pi_1(X)$ is trivial.

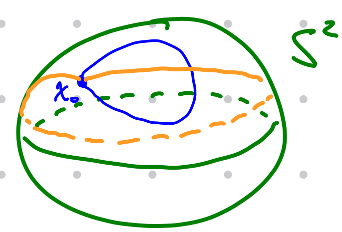
  ↳ contains only one element

  $\pi_1(X) \cong 1$

- Circle: $\pi_1(S^1) \cong \mathbb{Z}$

  ↑ additive group of integers

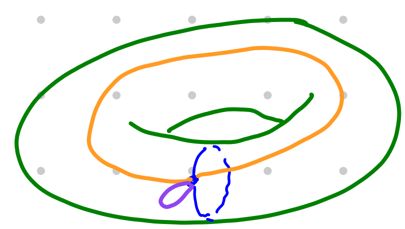
- Sphere: $\pi_1(S^2) \cong 1$ is trivial



Any space whose fundamental group is trivial is said to be *simply connected*.

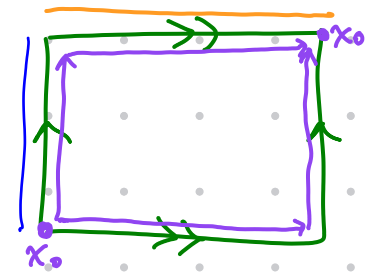
- Torus: $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}^2$

  (commutative group)



If X and Y are path-connected spaces, then

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

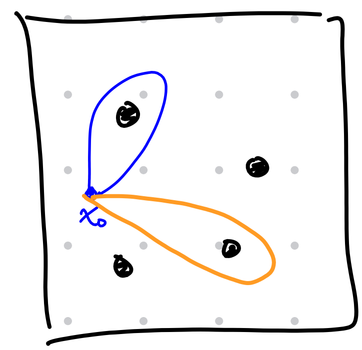


- Let X be a plane with k holes removed.

  $\pi_1(X)$ is a free group on k generators (non-commutative group)

  elements: strings of letters $a_1, a_2, \ldots, a_k$

  such that $a_1 a_1^{-1} = 1$, $a_2 a_2^{-1} = 1, \ldots$



k holes in this plane

There are also higher-order homotopy groups:

$$\pi_2(X), \quad \pi_3(X), \quad \pi_4(X), \ldots$$

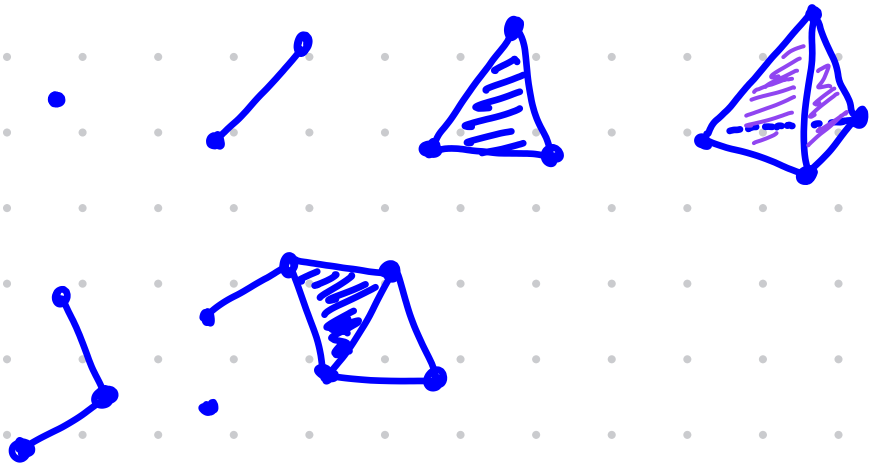
where $\pi_n(X)$ is the group of homotopy classes of basepoint-preserving maps $S^n \to X$.

in the Crossley text: Chapter 8

SIMPLICIAL COMPLEXES

[Chapter 7 in Crossley]

examples:

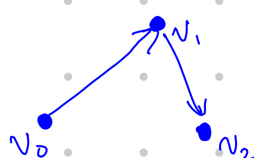


Definitions: A set of points $v_0, v_1, \ldots, v_k \in \mathbb{R}^n$ is in general position if the vectors

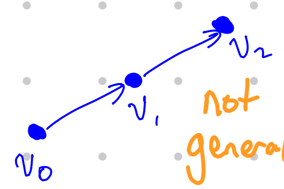
$$v_1 - v_0, \quad v_2 - v_1, \quad \ldots \quad v_k - v_{k-1}$$

are linearly independent.

or: "affinely independent"



in general position



not in general position

A k-simplex is the set of all convex combinations of k+1 points in general position in $\mathbb{R}^n$.

notation: $$\underbrace{[v_0, v_1, \ldots, v_k]}_{\text{k-simplex}} = \left\{ \lambda_0 v_0 + \cdots + \lambda_k v_k \;\middle|\; 0 \le \lambda_i \le 1 \text{ and } \sum_{i=0}^{k} \lambda_i = 1 \right\}$$

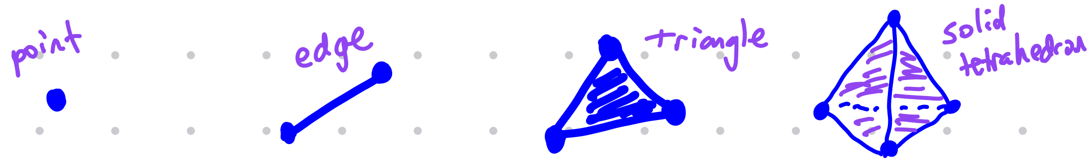
The points $v_0, v_1, \ldots, v_k$ are the vertices of the simplex

$$x = \lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2$$

for some nonnegative $\lambda_0, \ldots, \lambda_2$ with $\lambda_0 + \lambda_1 + \lambda_2 = 1$

convex combination

Examples:



0-simplex   1-simplex   2-simplex   3-simplex

A subset of vertices of a simplex forms a <u>subsimplex</u>

example: $S = [v_0, v_1, v_2, v_3]$ has subsimplices including

order of writing vertices doesn't matter

$[v_1]$, $[v_0, v_2]$, $[v_0, v_3]$, $[v_0, v_2, v_3]$, ...

face of S

A subsimplex that omits only one vertex is a <u>face</u>.

Simplicial Complexes

MATH 348

1. Suppose the points $v_0, \ldots, v_k \in \mathbb{R}^n$ are in general position. Show that

$$\{v_1 - v_0, \quad v_2 - v_0, \quad \ldots, \quad v_k - v_0\}$$

is a linearly independent set of vectors.

By the definition of general position, vectors $w_1 = v_1 - v_0$, $w_2 = v_2 - v_1$, .., $w_k = v_k - v_{k-1}$ are linearly independent

This means the matrix M whose rows are $w_1, w_2, .., w_k$ has rank k.

Let $u_1 = v_1 - v_0$, $u_2 = v_2 - v_0$, ..., $u_k = v_k - v_0$, and let matrix N have rows $u_1, u_2, .., u_k$.

Since $u_j = w_1 + w_2 + \cdots + w_j$, the matrix M can be transformed into N by elementary row operations.

Thus $\operatorname{rank}(M) = \operatorname{rank}(N) = k$, so the vectors $u_1, u_2, .., u_k$ are linearly independent.

2. Let $\sigma = [v_0, \ldots, v_k]$ be a simplex in $\mathbb{R}^n$. Let τ be a simplex whose vertices are some subset of $\{v_0, \ldots, v_k\}$. Show that $\tau \subseteq \sigma$.

By definition, $\sigma = [v_0, .., v_k] = \left\{ \lambda_0 v_0 + \cdots + \lambda_k v_k \;\middle|\; 0 \le \lambda_i \le 1 \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\}$.

Let $\{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}$ be a subset of $\{v_0, v_1, \ldots, v_k\}$. (This means $m \le k$.)

Then $\tau = [v_{i_1}, v_{i_2}, \ldots, v_{i_m}] = \left\{ \alpha_1 v_{i_1} + \cdots + \alpha_m v_{i_m} \;\middle|\; 0 \le \alpha \le 1 \text{ and } \sum_{i=1}^{m} \alpha_i = 1 \right\}$.

Let $x \in \tau$. Then $x = \alpha_{i_1} v_{i_1} + \cdots + \alpha_{i_m} v_{i_m}$. This means $x = \lambda_0 v_0 + \cdots + \lambda_k v_k$ where $\lambda_j = \alpha_j$ if $j \in \{i_1, .., i_m\}$ and $\lambda_j = 0$ otherwise. Thus $x \in \sigma$.

Therefore, $\tau \subseteq \sigma$

3. Show that an n-simplex is homeomorphic to a closed n-dimensional ball.

An n-simplex has $n+1$ vertices. As $n+1$ points in general position, these vertices determine an $(n-1)$-sphere (the "circumsphere" of the simplex)

This $(n-1)$-sphere, together with its interior, is a closed n-ball.

Map each face of the simplex onto this sphere, expanding the interior of the simplex continuously.

This gives a continuous map from the n-simplex to the closed n-ball with continuous inverse.

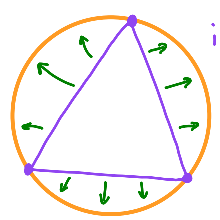


illustration for $n = 3$

4. Let σ be a k-simplex.

   (a) How many subsimplices does σ have?

   $2^{k+1} - 1$ subsimplices $\longleftarrow$ These correspond to all nonempty subsets of $\{v_0, v_1, v_2, \ldots, v_k\}$

   (b) How many faces does σ have?

   $k+1$ faces $\longleftarrow$ Omit exactly one vertex from $\{v_0, v_1, \ldots, v_k\}$ to obtain a face

We did not get to these problems in class.

5. Describe the boundary of a simplex in terms of barycentric coordinates.

6. Consider the three conditions in the definition of a simplicial complex. Give an example of a subspace K of $\mathbb{R}^n$ and a finite list of simplices that satisfies:

   (a) Properties 1 and 2, but not 3.

   (b) Properties 1 and 3, but not 2.

   (c) Properties 2 and 3, but not 1.

7. Let S and T be simplices of a simplicial complex K. Show that $S \cap T$ is either empty or a subsimplex of both S and T.