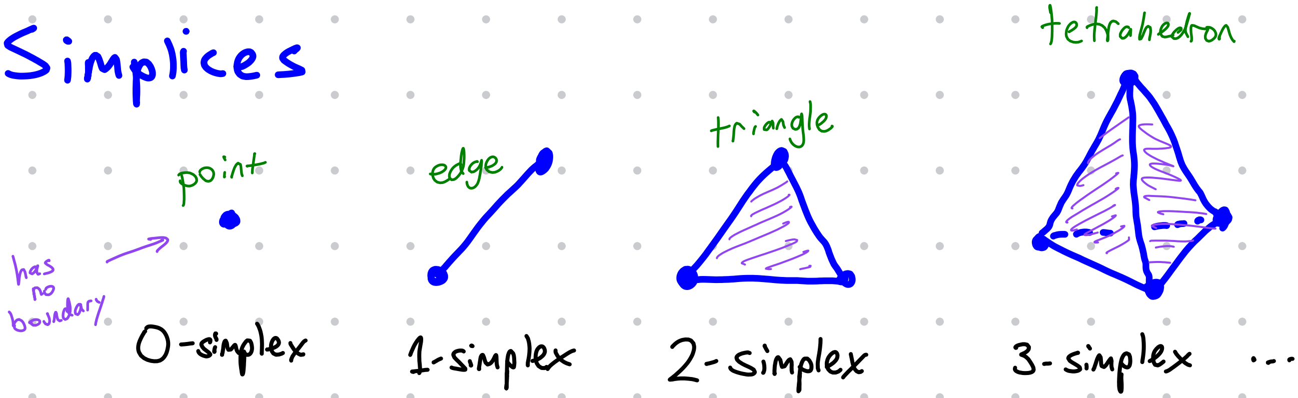
14 November 2024

Simplices

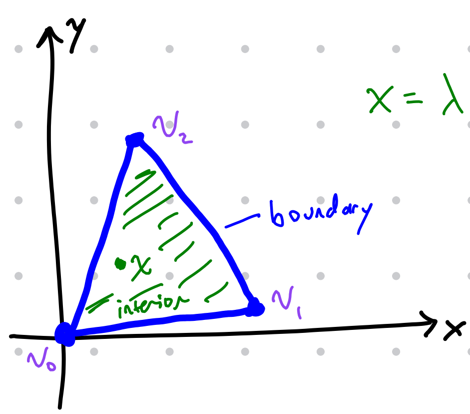


definition:

$$[v_0, \dots, v_k] = \left\{ \lambda_0 v_0 + \dots + \lambda_k v_k \;\middle|\; 0 \le \lambda_i \le 1 \text{ and } \sum_{i=0}^{k} \lambda_i = 1 \right\}$$

Coefficients $\lambda_0, \lambda_1, \dots, \lambda_k$ are called the barycentric coordinates of the point $x \in [v_0, \dots, v_k]$.

example:



$x = \lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2$

where $0 \le \lambda_i$ and $\lambda_0 + \lambda_1 + \lambda_2 = 1$

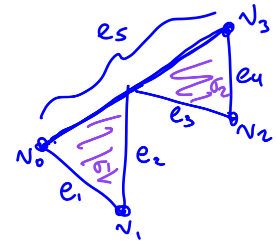
A point x is on the boundary of a simplex if at least one of its barycentric coordinates is zero. Otherwise, x is in the interior.

5. Describe the boundary of a simplex in terms of barycentric coordinates.

6. Consider the three conditions in the definition of a simplicial complex. Give an example of a subspace K of $\mathbb{R}^n$ and a finite list of simplices that satisfies:

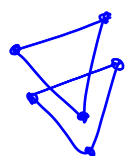
(a) Properties 1 and 2, but not 3.

open triangle?

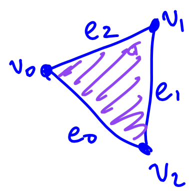


list: $\{v_0, v_1, v_2, v_3, e_1, e_2, e_3, e_4, e_5, \sigma_1, \sigma_2\}$

(b) Properties 1 and 3, but not 2.



(c) Properties 2 and 3, but not 1.

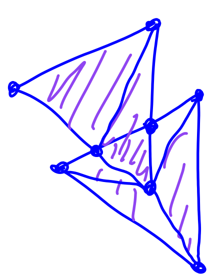


$\{v_0, v_1, v_2, e_0, e_1, e_2, \sigma\}$

K = boundary of the triangle

7. Let S and T be simplices of a simplicial complex K. Show that $S \cap T$ is either empty or a subsimplex of both S and T.

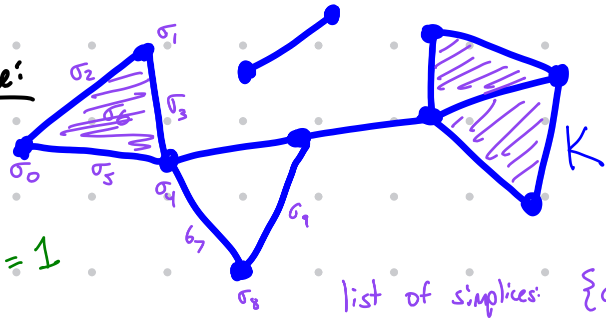
Proposition 7.4

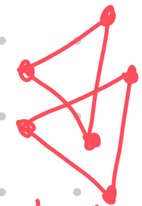


Simplicial Complex:

A collection of simplices that satisfies certain intersection properties

example:





not allowed

$i_0 = 11$
$i_1 = 13$
$i_2 = 3$
$\chi(K) = 11 - 13 + 3 = 1$

list of simplices: $\{\sigma_0, \sigma_1, \dots, \sigma_N\}$

Definition: A simplicial complex K is a subspace of $\mathbb{R}^n$ with a finite list of simplices such that:

1. The union of the simplices is K.
2. Each point of K lies in the interior of exactly one simplex.
3. Every face of (every simplex in the list) is also in the list.

Definition: If T is an n-dimensional simplicial complex and i_k is the number of k-simplices for each k, then the Euler characteristic (or Euler number) is

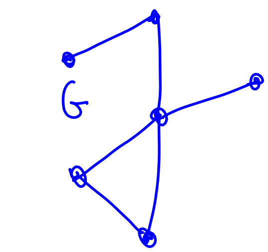
$$\chi(T) = \sum_{k=1}^{n} (-1)^k i_k = i_0 - i_1 + i_2 - i_3 + \dots + (-1)^n i_n .$$

chi

Euler Characteristic

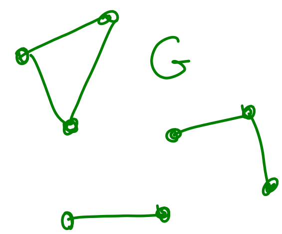
MATH 348

1. Draw several planar graphs and compute their Euler characteristics. What does the Euler characteristic tell you about a planar graph?



$i_0 = 6$, $i_1 = 6$

$\chi(G) = 6 - 6 = 0$



$\chi(G) = 8 - 6 = 2$

number of connected pieces minus the number of loops

2. Extend problem #1 to planar two-dimensional simplicial complexes. What does the Euler characteristic tell you about such complexes?

3. Explain why the Euler characteristic of a simplicial complex K can also be written

$$\chi(K) = \sum_{\sigma \in K} (-1)^{\dim(\sigma)}$$

where the sum is over all simplices σ in K, and $\dim(\sigma)$ is the dimension of σ.

4. Compute $\chi(S^2)$ using at least two different triangulations of the sphere S^2.

5. Compute $\chi(T^2)$ using at least two different triangulations of the torus T^2.