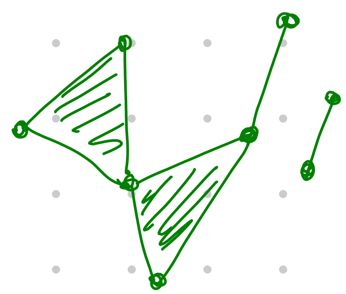
19 November 2024

Simplicial Complex: collection of simplices such that the intersection of any two simplices is a subsimplex of each.

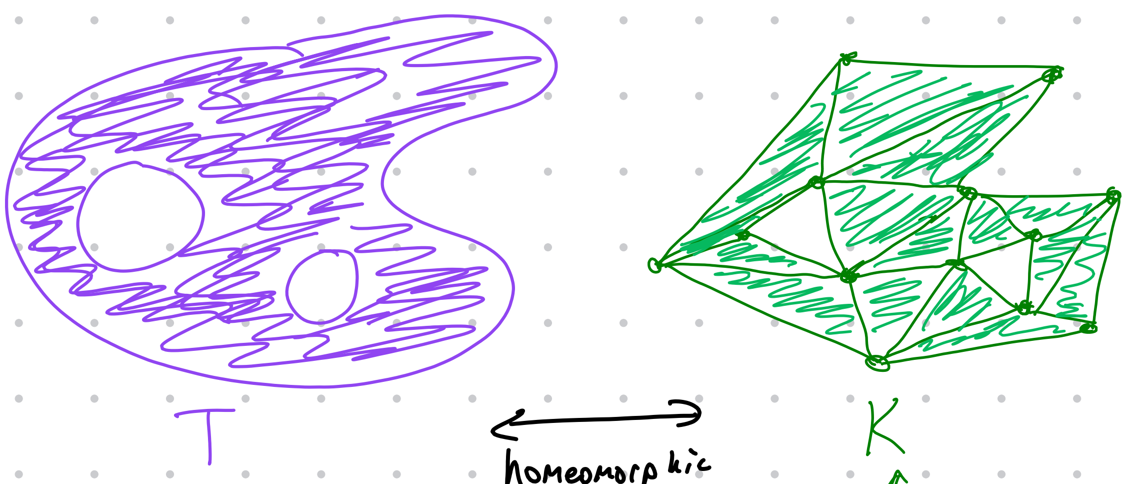


Euler Characteristic: $\sum_{k=0}^{N} (-1)^k i_k$

where i_k is the number of k-simplices

(# vertices) - (# edges) + (# faces) - (# tetrahedra) + ...

Triangulation: of a topological space T is a simplicial complex K and a homeomorphism $K \leftrightarrow T$.



↑ triangulation of T

<u>Fact</u>: Any two triangulations of T have the same Euler char.

<u>Note</u>: if T has a triangulation, then T must be closed

Euler Characteristic

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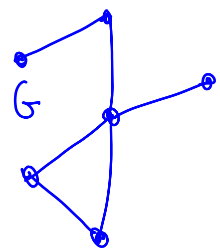
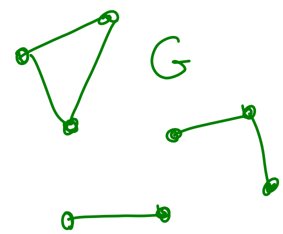
1. Draw several planar graphs and compute their Euler characteristics. What does the Euler characteristic tell you about a planar graph?



$i_0 = 6$, $i_1 = 6$

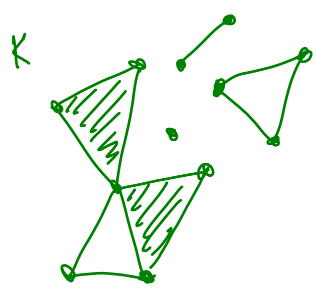
$\chi(G) = 6 - 6 = 0$



$\chi(G) = 8 - 6 = 2$

number of connected pieces minus the number of loops

2. Extend problem #1 to planar two-dimensional simplicial complexes. What does the Euler characteristic tell you about such complexes?

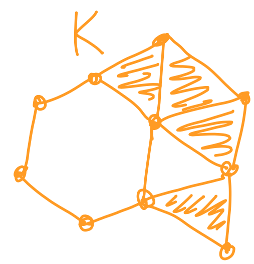


$\chi(K) = \underset{\text{vertices}}{12} - \underset{\text{edges}}{12} + \underset{\text{faces}}{2} = 2$

4 connected pieces

2 loops

$4 - 2 = 2$



$\chi(K) = \underset{\text{vertizes}}{10} - \underset{\text{edges}}{15} + \underset{\text{faces}}{4} = -1$

1 connected piece

2 loops

$1 - 2 = -1$

$\chi(K)$ = number of connected pieces − number of loops

3. Explain why the Euler characteristic of a simplicial complex K can also be written

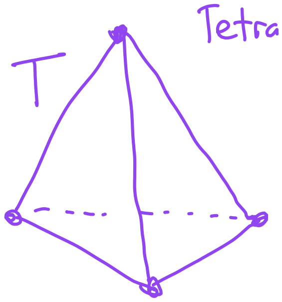
$$\chi(K) = \sum_{\sigma \in K} (-1)^{\dim(\sigma)}$$

where the sum is over all simplices σ in K, and $\dim(\sigma)$ is the dimension of σ.

4. Compute $\chi(S^2)$ using at least two different triangulations of the sphere S^2.

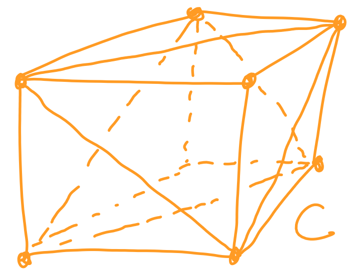
$\chi(S^2) = 2$

hollow Tetrahedron



$\chi(T) = 4 - 6 + 4$
$= 2$

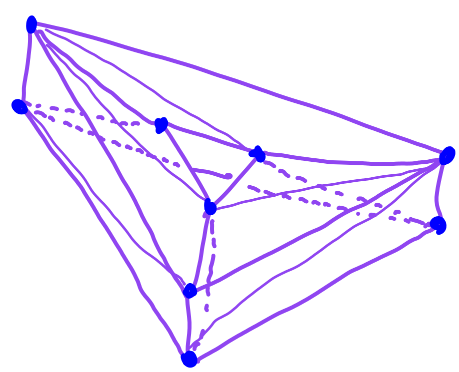
triangulated cube



$\chi(C) = 8 - 18 + 12$
$= 2$

5. Compute $\chi(T^2)$ using at least two different triangulations of the torus T^2.

$\chi(T^2) = 0$



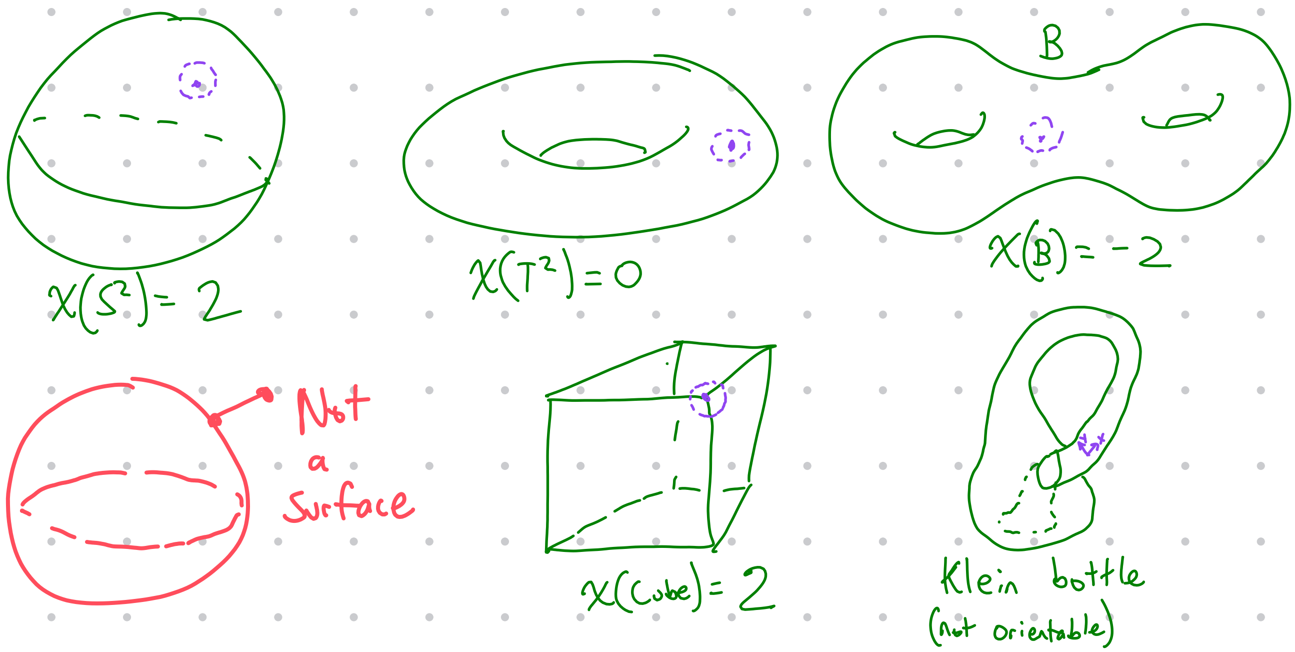
9 vertices

27 edges

18 faces

$\chi(T^2) = 9 - 27 + 18$
$= 0$

Surface: A connected Housdorff space with the property that around any point, there is an open neighborhood homeomorphic to $\mathbb{R}^2$



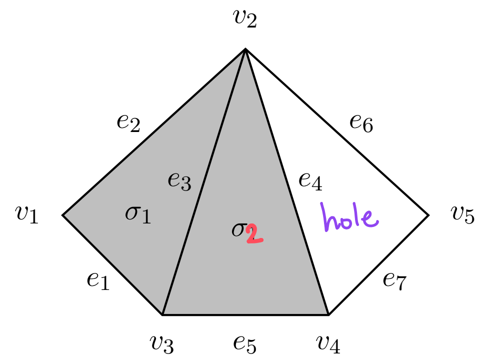
Classification of surfaces

Two triangulable surfaces are homeomorphic iff they have the same Euler characteristic and orientability.

Introduction to Simplicial Homology

MATH 348

Consider the following simplicial complex K. Note that K consists of two 2-simplices, seven 1-simplices, and five 0-simplices. Moreover, K has one *hole*—the triangle that is not filled in.*



1. Write three one-dimensional loops that *do not* encircle the hole. Write each loop as a sum of 1-simplices.

$\gamma_1 = \underbrace{e_3 + e_4 + e_5}_{\text{"formal sum" of 1-simplices}}$

$\gamma_2 = e_1 + e_2 + e_3$

$\gamma_3 = e_1 + e_2 + e_4 + e_5$

2. Let γ_1 and γ_2 be loops that you wrote in #1. Consider the difference $\gamma_1 - \gamma_2$ with $\mathbb{Z}_2$ coefficients. What can you say about this difference?

also denoted $\mathbb{Z}/2$

$\gamma_1 - \gamma_2 = (e_3 + e_4 + e_5) - (e_1 + e_2 + e_3) = e_4 + e_5 - 1e_1 - 1e_2 = e_1 + e_2 + e_4 + e_5$

3. Write three one-dimensional loops that encircle the hole. Write each loop as a sum of 1-simplices.

to be continued..

$e_1 + e_2 + e_5 + e_6 + e_7$

4. Let γ_1 and γ_2 be loops that you wrote in #3. Consider the difference $\gamma_1 - \gamma_2$ with $\mathbb{Z}_2$ coefficients. What can you say about this difference?

*source: Michael Starbird and Francis Su, *Topology Through Inquiry*, Section 15.2.