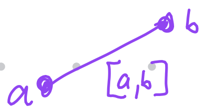
21 November 2024

How do we detect a "hole" in a topological space?

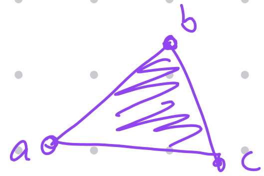
→ Find an object that surrounds the hole.

The boundary operator δ takes each simplex to the sum of its faces, and extends linearly to sums of simplices.

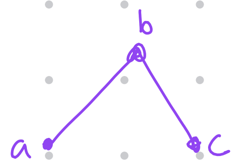
examples:



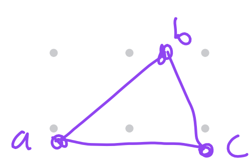
$\delta[a,b] = a+b$



$\delta[a,b,c] = \underbrace{[a,b] + [b,c] + [a,c]}_{\text{a 1-chain}}$



$$\begin{aligned}\delta\big([a,b]+[b,c]\big) &= \delta[a,b] + \delta[b,c] \\ &= a+b + b+c \\ &= a+c \leftarrow \text{a 0-chain}\end{aligned}$$



$$\begin{aligned}\delta\big([a,b]+[b,c]+[a,c]\big) &= \delta[a,b]+\delta[b,c]+\delta[a,c] \\ &= a+b+b+c+a+c \\ &= 0\end{aligned}$$

A n-chain is a sum of n-simplices.

An n-cycle is an n-chain whose boundary is zero.

An n-boundary is an n-chain that is the boundary of an (n+1)-chain

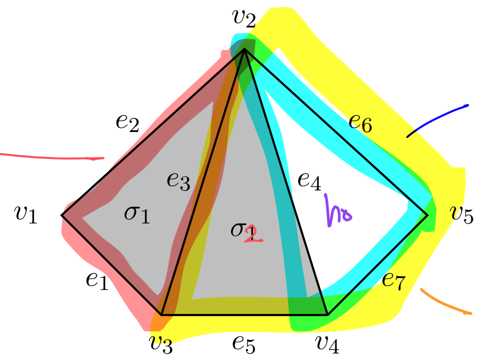
INTUITION: a hole is a cycle that is not a boundary

Introduction to Simplicial Homology

MATH 348

Consider the following simplicial complex K. Note that K consists of two 2-simplices, seven 1-simplices, and five 0-simplices. Moreover, K has one *hole*—the triangle that is not filled in.*

cycle $e_1+e_2+e_3$ is the boundary of σ_1



cycle $e_4+e_6+e_7$ is not the boundary of any 2-chain

cycle $e_3+e_5+e_6+e_7$ is somehow going around the same hole as $e_4+e_6+e_7$

1. Write three one-dimensional loops that *do not* encircle the hole. Write each loop as a sum of 1-simplices.

$\gamma_1 = e_1+e_2+e_3$

$\gamma_2 = e_3+e_4+e_5$

$\gamma_3 = e_1+e_2+e_4+e_5$

examples of 1-cycles that are boundaries

2. Let γ_1 and γ_2 be loops that you wrote in #1. Consider the difference $\gamma_1 - \gamma_2$ with $\mathbb{Z}_2$ coefficients. What can you say about this difference?

mod 2: $-1=1$

$\gamma_1+\gamma_2 = e_1+e_2+\underbrace{\cancel{2e_3}}_{0}+e_4+e_5 = \gamma_3$

$\gamma_1-\gamma_2 = e_1+e_2+e_3-(e_3+e_4+e_5) = e_1+e_2-\cancel{1}e_4-\cancel{1}e_5$

3. Write three one-dimensional loops that encircle the hole. Write each loop as a sum of 1-simplices.

$e_4+e_6+e_7$

$e_1+e_2+e_3+e_4+e_6+e_7$

$\gamma_1 = e_1+e_2+e_5+e_6+e_7$

$\gamma_2 = e_3+e_5+e_6+e_7$

examples of 1-cycles

4. Let γ_1 and γ_2 be loops that you wrote in #3. Consider the difference $\gamma_1 - \gamma_2$ with $\mathbb{Z}_2$ coefficients. What can you say about this difference?

$\gamma_1-\gamma_2 = (e_1+e_2+\cancel{e_5}+\cancel{e_6}+\cancel{e_7}) - (e_3+\cancel{e_5}+\cancel{e_6}+\cancel{e_7})$

$= e_1+e_2+e_3 = \delta(\sigma_1)$

$\gamma_1-\gamma_2$ is the boundary of some 2-chain

*source: Michael Starbird and Francis Su, *Topology Through Inquiry*, Section 15.2.

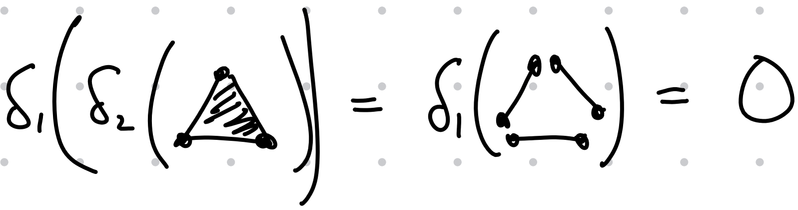
<u>Important property:</u> The boundary of a boundary is zero.

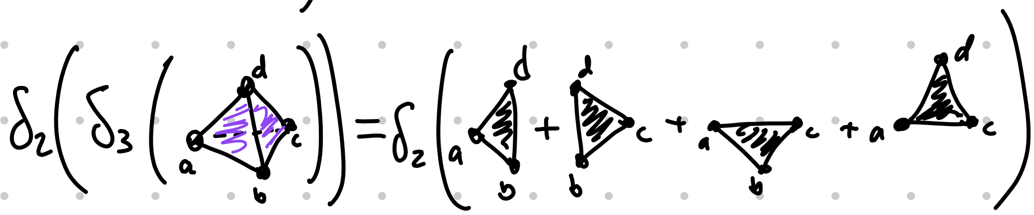
$$\delta_n \circ \delta_{n+1} = 0$$

boundary of an n-chain

boundary of an (n+1)-chain

<u>illustration:</u>





$$= \Big(\cancel{[a,b]} + \cancel{[a,d]} + \cancel{[b,d]}\Big) + \Big(\cancel{[b,d]} + \cancel{[b,c]} + \cancel{[c,d]}\Big)$$
$$+ \Big(\cancel{[a,b]} + \cancel{[b,c]} + \cancel{[a,c]}\Big) + \Big(\cancel{[a,c]} + \cancel{[a,d]} + \cancel{[c,d]}\Big)$$

$$= 0$$

<u>NOTATION:</u>

have no boundary

$Z_n(K)$ is the <u>set</u> of n-cycles of simplicial complex K

$B_n(K)$ is the <u>set</u> of n-boundaries of simplicial complex K

are the boundary of something

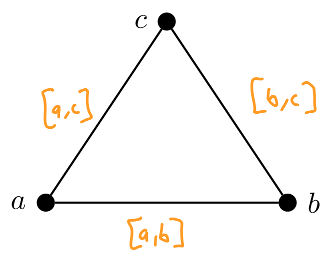
$C_n(K)$ is the <u>set</u> of all n-chains of K

alternatively, vector space

Introduction to Simplicial Homology

MATH 348

1. Consider the following simplicial complex K.



(a) What are C_0 and C_1? What are their dimensions?

C_0: set or vector space of all 0-chains: basis for $C_0 = \{a, b, c\}$

$\dim(C_0) = 3$

C_1: set or vector space of all 1-chains: basis for $C_1 = \{[a,b], [b,c], [a,c]\}$

$\dim(C_1) = 3$

(b) What is the dimension of Z_1? (Hint: What is the only nontrivial 1-cycle?)

1-chains that have no boundary

$Z_1 = \{[a,b] + [b,c] + [a,c]\}$ $\quad$ $\dim(Z_1) = 1$

(c) What is B_1?

There are no 1-chains that are boundaries.

so $\dim(B_1) = 0$

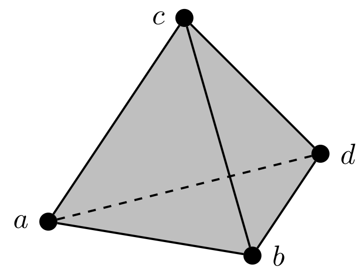
(d) What vector space is $H_1(K)$? What is its dimension?

(e) What is the dimension of $\operatorname{Im} \delta_1$? (Hint: rank-nullity theorem)

(f) What vector space is $H_0(K)$? What is its dimension?

(g) Why is $H_i(K) = 0$ for all $i > 1$?

2. Consider the following simplicial complex K, which is a *hollow tetrahedron.*



(a) What are the dimensions of C_i for all nonnegative integers i?

(b) What is the dimension of $\operatorname{Im}\delta_1$?

(c) What is the dimension of $H_0(K)$?

(d) What are the dimensions of $\operatorname{Ker}\delta_1$ and $\operatorname{Im}\delta_2$?

(e) What is the dimension of $H_1(K)$?

(f) What are the dimensions of $\operatorname{Ker}\delta_2$ and $\operatorname{Im}\delta_3$?

(g) What is the dimension of $H_2(K)$?