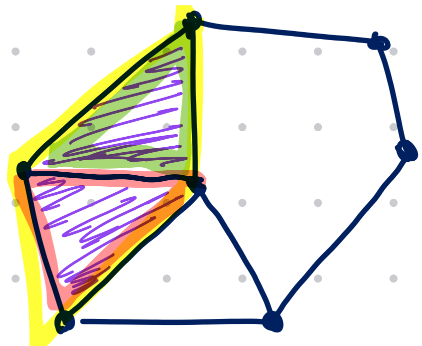
26 November 2024

K



Homology counts (equivalence classes of) cycles that are not boundaries.

cycle: a collection of n-simplices that has no boundary

boundary: of an n-simplex is the set of its $(n-1)$-dimensional faces

$C_n(K)$ is the vector space of n-chains of K, which are finite sums $\sigma_1 + \sigma_2 + \cdots + \sigma_k$ of distinct n-simplices of K.

Boundary operator: $\delta_n : C_n(K) \to C_{n-1}(K)$
takes each simplex to its boundary

NOTE:
$$\delta_n \circ \delta_{n+1} = 0$$

The boundary of a boundary is zero.

$Z_n(K)$ is the vector space of n-cycles

NOTE: $Z_n(K) = \mathrm{Ker}(\delta_n)$

$B_n(K)$ is the vector space of n-boundaries, which are n-chains that are the boundary of $(n+1)$-chains

NOTE: $B_n(K) = \mathrm{Im}(\delta_{n+1})$

The $n^{th}$ homology ~~group~~ vector space is

$$H_n(K) = \frac{Z_n(K)}{B_n(K)}$$

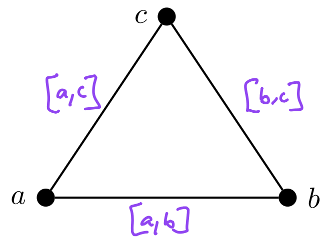
QUOTIENT VECTOR SPACE

Often, we mostly care about the dimension of $H_n(K)$, which is called the $n^{th}$ Betti number of K.

# Introduction to Simplicial Homology

MATH 348

**1.** Consider the following simplicial complex $K$.




(a) What are $C_0$ and $C_1$? What are their dimensions?

$C_0$ basis $\{a, b, c\}$, $\dim C_0 = 3$

$C_1$ basis $\{[a,b], [a,c], [b,c]\}$, $\dim C_1 = 3$

$\dim C_2 = 0$

$\vdots$

(b) What is the dimension of $Z_1$? (Hint: What is the only nontrivial 1-cycle?)

1-cycles

Only nontrivial loop: $[a,b] + [b,c] + [a,c]$ $\qquad \dim Z_1 = 1$

(c) What is $B_1$?

1-boundaries

none $\qquad \dim B_1 = 0$

(d) What vector space is $H_1(K)$? What is its dimension?

$$H_1 = \frac{Z_1}{B_1} = Z_1 = \mathbb{Z}/2$$

1-dimensional vector space with mod 2 coefficients

$\dim(H_1) = 1$ — 1st Betti number of K

INTERPRETATION: the number of holes in K

(e) What is the dimension of $\operatorname{Im}\delta_1$? (Hint: rank-nullity theorem)

$\delta_1 : C_1 \longrightarrow C_0$

domain: $C_1$ has dimension 3

$\dim \operatorname{Ker}(\delta_1) = 1$

$$\dim(C_1) = \dim \operatorname{Ker}(\delta_1) + \dim \operatorname{Im}(\delta_1)$$
$$3 = 1 + 2$$

$B_0 = (\mathbb{Z}/2)^2$

(f) What vector space is $H_0(K)$? What is its dimension?

$$H_0 = \frac{Z_0}{B_0} = \frac{C_0}{B_0} = \mathbb{Z}/2$$

$C_0 \leftarrow \dim 3$

$B_0 = \operatorname{Im}\delta_1 \qquad B_0 \uparrow \dim 2$

$\dim(H_0) = 3 - 2 = 1$ — 0th Betti num. of K

INTERPRETATION: K has 1 connected component

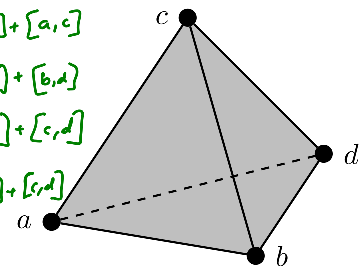
(g) Why is $H_i(K) = 0$ for all $i > 1$?

$$H_i = \frac{Z_i}{B_i} = 0$$

since $Z_i = 0$, since there are no cycles of dimension $> 1$ in K

**2.** Consider the following simplicial complex $K$, which is a *hollow tetrahedron.*

$\delta_2([a,b,c]) = [a,b] + [b,c] + [a,c]$
$\delta_2([a,b,d]) = [a,b] + [a,d] + [b,d]$
$\delta_2([a,c,d]) = [a,c] + [a,d] + [c,d]$
$\delta_2([b,c,d]) = [b,c] + [b,d] + [c,d]$




$\delta_1([a,b]) = a+b$
$\delta_1([a,c]) = a+c$
$\delta_1([a,d]) = a+d$
$\delta_1([b,c]) = b+c$
$\delta_1([b,d]) = b+d$
$\delta_1([c,d]) = c+d$

(a) What are the dimensions of $C_i$ for all nonnegative integers $i$?

$\dim C_0 = 4$ $\qquad \dim C_2 = 4$ $\qquad$ also $\dim C_i = 0$ for $i > 2$.

$\dim C_1 = 6$ $\qquad \dim C_3 = 0$

(b) What is the dimension of $\operatorname{Im} \delta_1$?

$\delta_1 = C_1 \to C_0$

$\dim(\operatorname{Im} \delta_1) = 3 = \dim(B_0)$

since there are exactly 3 lin. indep vectors in $\operatorname{Im} \delta_1$

(c) What is the dimension of $H_0(K)$?

$H_0 = \dfrac{Z_0}{B_0}$

$\dim(H_0) = \dim(Z_0) - \dim(B_0) = \dim(C_0) - \dim(B_0)$
$= 4 - 3 = 1$

K has 1 connected component

Note: $Z_0 = C_0$ since vertices have no boundary

(d) What are the dimensions of $\operatorname{Ker} \delta_1$ and $\operatorname{Im} \delta_2$?

$\dim(C_1) = \dim(\operatorname{Im} \delta_1) + \dim(\operatorname{Ker} \delta_1)$
$6 = 3 + 3$

$\dim(\operatorname{Im} \delta_2) = 3$

(e) What is the dimension of $H_1(K)$?

$H_1 = \dfrac{Z_1}{B_1}$

$\dim(H_1) = \dim(Z_1) - \dim(B_1)$
$= \dim(\operatorname{Ker} \delta_1) - \dim(\operatorname{Im} \delta_2) = 3 - 3 = 0$

K has no loops

(f) What are the dimensions of $\operatorname{Ker} \delta_2$ and $\operatorname{Im} \delta_3$?

$\dim(C_2) = \dim(\operatorname{Im}(\delta_2)) + \dim(\operatorname{Ker} \delta_2)$
$4 = 3 + 1$

$\dim(\operatorname{Im} \delta_3) = 0$

(g) What is the dimension of $H_2(K)$?

$H_2 = \dfrac{Z_2}{B_2}$

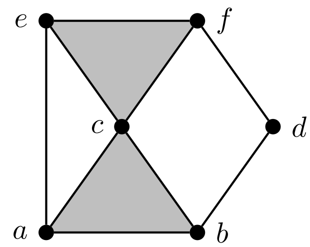
$\dim(H_2) = \dim(Z_2) - \dim(B_2)$
$= \dim(\operatorname{Ker} \delta_2) - \dim(\operatorname{Im} \delta_3) = 1 - 0 = 1$

K has one void

# Computing Simplicial Homology

MATH 348

Consider the following simplicial complex $K$.




**1.** What are the dimensions of $C_0$, $C_1$, and $C_2$?

$\dim(C_0) = 6$ $\qquad$ $\dim(C_1) = 9$ $\qquad$ $\dim(C_2) = 2$

**2.** Write a matrix representing the map $\delta_1$.

$\delta_1 : C_1 \to C_0$

$$\delta_1 : \begin{array}{c|ccccccccc} & ab & ac & ae & bc & bd & ce & cf & df & ef \\ \hline a & 1 & 1 & 1 & & & & & & \\ b & 1 & & & 1 & 1 & & & & \\ c & & 1 & & 1 & & 1 & 1 & & \\ d & & & & & 1 & & & 1 & \\ e & & & 1 & & & 1 & & & 1 \\ f & & & & & & & 1 & 1 & 1 \end{array}$$

$\operatorname{rank}(\delta_1) = 5$

**3.** What is the dimension of $H_0(K)$?

$H_0 = \dfrac{Z_0}{B_0}$

$\dim H_0 = \dim(C_0) - (\operatorname{rank} \delta_1)$

$= 6 - 5 = 1$

number of connected components of $K$

**4.** Write a matrix representing the map $\delta_2$.

$$\delta_2 : \begin{array}{c|cc} & abc & cef \\ \hline ab & 1 & \\ ac & 1 & \\ ae & & \\ bc & 1 & \\ bd & & \\ ce & & 1 \\ cf & & 1 \\ df & & \\ ef & & 1 \end{array}$$

$\operatorname{rank}(\delta_2) = 2$

**5.** What is dimension of $H_1(K)$?

number of holes in $K$

$\dim H_1 = \dim(\ker \delta_1) - \operatorname{rank} \delta_2 = 4 - 2 = 2$

**6.** What are the dimensions of $H_n(K)$ for $n > 1$?