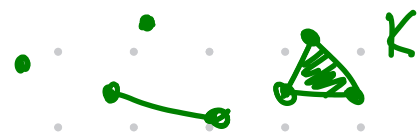
3 December 2024

Homology counts holes in a simplicial complex

different typles of holes:

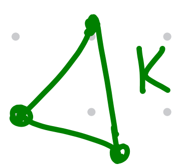
- 0-dimensional holes: connected components

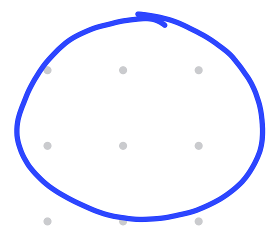


dimension-0 homology vector space

0th Betti number $\rightarrow$ $\beta_0(K)$ = number of connected components = $\dim(H_0(K))$

- 1-dimensional holes: what you usually think of as holes



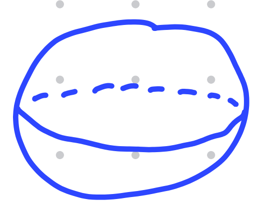


1st Betti number $\rightarrow$ $\beta_1(K)$ = number of 1-dimensional holes = $\dim(H_1(K))$

Homology vector space in dim 1

- 2-dimensional holes: voids enclosed by a surface

hollow tetrahedron



hollow sphere

2nd Betti number $\rightarrow$ $\beta_2(K)$ = number of 2-dimensional holes = $\dim(H_2(K))$

dim-2 homology vector space

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def: $H_j(K) = \dfrac{Z_j(K)}{B_j(K)}$ $\leftarrow$ cycles (have no boundary) / $\leftarrow$ boundaries

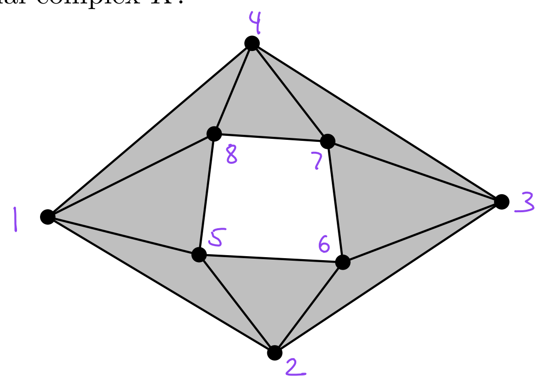
δ_j : j-dimensional boundary operator

$$\dim(H_j) = \dim(Z_j) - \dim(B_j) = \text{nullity}(\delta_j) - \text{rank}(\delta_{j+1})$$

Computing Simplicial Homology

MATH 348

Consider the following simplicial complex K.



δ_0: 8 columns

0 rows

nullity $(\delta_0) = 8$

1. Write down matrices representing the boundary maps δ_1 and δ_2 for K.

2. Reduce your matrices by left-to-right column additions using arithmetic mod 2.

3. Compute $\dim H_0(K)$ and $\dim H_1(K)$.

$$\delta_1: \begin{array}{c|cccccccccccccccc|}
 & 12 & 14 & 15 & 18 & 23 & 25 & 26 & 34 & 36 & 37 & 47 & 48 & 56 & 58 & 67 & 78 \\
\hline
1 & 1 & 1 & 1 & 1 & & 0 & & 0 & & & 0 & 0 & 0 & 0 & & 0 \\
2 & \boxed{1} & & & & 1 & 0 & 1 & 0 & 0 & & 0 & & 0 & & 0 & 0 \\
3 & & & & & \boxed{1} & & & 0 & 0 & 1 & 0 & & & & & 0 \\
4 & & \boxed{1} & & & & & & 0 & & & 0 & 0 & & & & \\
5 & & & \boxed{1} & & & 0 & & & & & & & 0 & 0 & & \\
6 & & & & & & & \boxed{1} & & 0 & & & & 0 & & 0 & \\
7 & & & & & & & & & & \boxed{1} & 0 & & & & 0 & 0 \\
8 & & & & \boxed{1} & & & & & & & & 0 & & 0 & & 0 \\
\hline
\end{array}$$

rank $(\delta_1) = 7$

nullity $(\delta_1) = 9$

$$\delta_2: \begin{array}{c|cccccccc|}
 & 125 & 148 & 158 & 236 & 256 & 347 & 367 & 478 \\
\hline
12 & 1 & & & & & & & \\
14 & & 1 & & & & & & \\
15 & 1 & & 1 & & & & & \\
18 & & 1 & 1 & & & & & \\
23 & & & & 1 & & & & \\
25 & \boxed{1} & & & & 1 & & & \\
26 & & & & 1 & 1 & & & \\
34 & & & & & & 1 & & \\
36 & & & & \boxed{1} & & & 1 & \\
37 & & & & & & 1 & 1 & \\
47 & & & & & & \boxed{1} & & 1 \\
48 & & \boxed{1} & & & & & & 1 \\
56 & & & & & \boxed{1} & & & \\
58 & & & \boxed{1} & & & & & \\
67 & & & & & & & \boxed{1} & \\
78 & & & & & & & & \boxed{1} \\
\hline
\end{array}$$

rank $(\delta_2) = 8$

num. vertices

$$\beta_0 = \dim(H_0) = \text{nullity}(\delta_0) - \text{rank}(\delta_1)$$
$$= 8 - 7 = 1$$

K has 1 connected component

$$\beta_1 = \dim(H_1) = \text{nullity}(\delta_1) - \text{rank}(\delta_2)$$
$$= 9 - 8 = 1$$

K has one hole

(1-dim'l hole)

$$\beta_2 = \dim(H_2) = \text{nullity}(\delta_2) - \text{rank}(\delta_3)$$
$$= 0 - 0 = 0$$

K has no enclosed volumes