5 December 2024

Recall Euler characteristic

alternating sum of the numbers of simplices

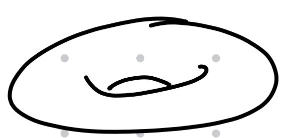
$$\chi(K) = i_0(K) - i_1(K) + \cdots + (-1)^n i_n(K) = \sum_{j=0}^{n} (-1)^j i_j(K)$$

where $i_j(K)$ is the number of j-dimensional simplices in complex K (say K is an n-dimensional complex)

<u>Theorem</u>: Euler characteristic is an alternating sum of the Betti numbers:

$$\chi(K) = \beta_0(K) - \beta_1(K) + \cdots + (-1)^n \beta_n(K) = \sum_{j=0}^{n} (-1)^j \beta_j(K)$$

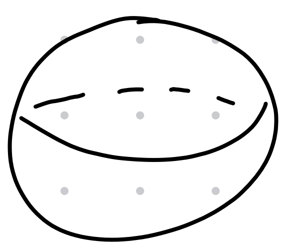
<u>Examples</u>: Torus T



$$\chi(T) = 0$$

$$\beta_0 - \beta_1 + \beta_2 = 1 - 2 + 1 = 0$$

Sphere S^1



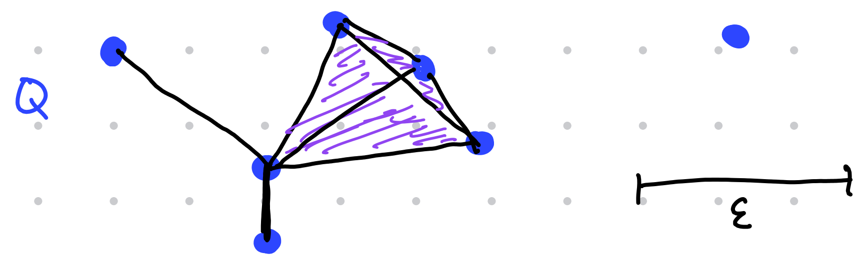
$$\chi(S^1) = 2$$

$$\beta_0 - \beta_1 + \beta_2 = 1 - 0 + 1 = 2$$

Setting: Let $Q \subset \mathbb{R}^n$ be a finite set of points, which we will call a point cloud.

Suppose Q is sampled from an unknown topological space X.

How can we recover the topology of X from Q?

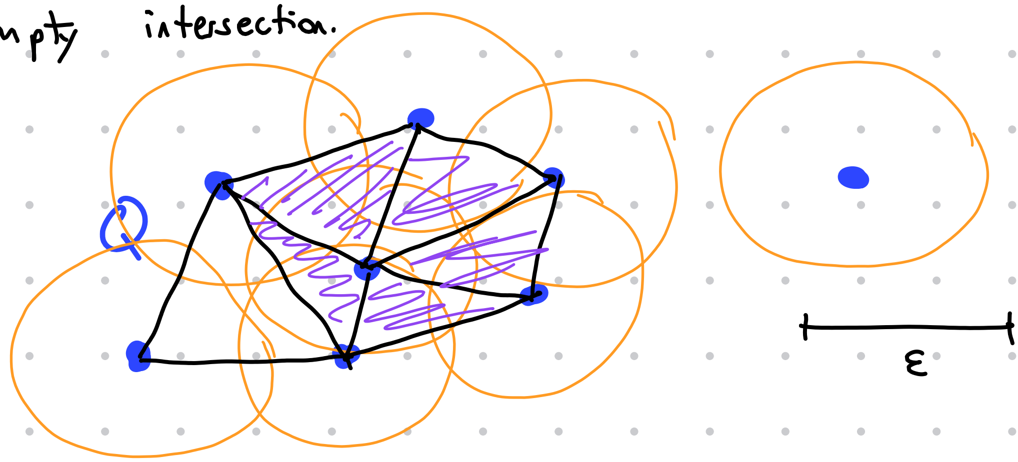


Vietoris-Rips Complex: Choose some $\varepsilon \geq 0$.

Then the Vietoris-Rips complex at scale ε, $VR_\varepsilon(Q)$ contains a simplex $[v_0, v_1, \ldots, v_k]$ exactly when $d(v_i, v_j) \leq \varepsilon$ for all $i, j \in \{0, 1, \ldots, k\}$.

Čech complex: Choose some $\varepsilon \geq 0$.

Then the Čech complex at scale ε, $\check{C}_\varepsilon(Q)$ contains a simplex $[v_0, v_1, \ldots, v_k]$ exactly when the balls of diameter ε centered at $v_0, v_1, \ldots, v_k$ have nonempty intersection.



Simplicial Complexes from Data

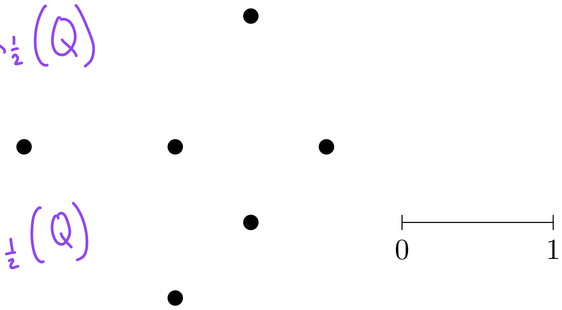
MATH 348

1. Consider the following point cloud in $\mathbb{R}^2$. For several $\epsilon > 0$ of your choice, construct the simplicial complexes $VR_\epsilon(Q)$ and $\check{C}_\epsilon(Q)$.

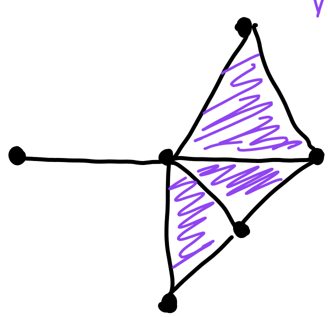
$\varepsilon = \frac{1}{2}$

$VR_{\frac{1}{2}}(Q)$

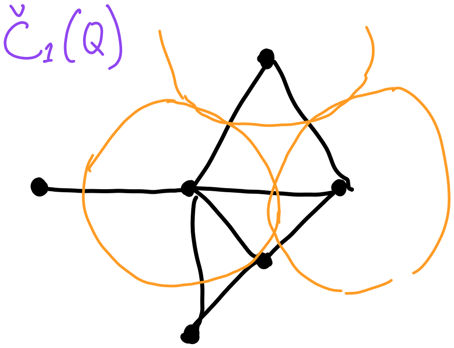
$\check{C}_{\frac{1}{2}}(Q)$



$VR_1(Q)$



$\check{C}_1(Q)$



2. Let Q be a set of points in $\mathbb{R}^n$.

(a) For what parameters ϵ and δ is $VR_\epsilon(Q)$ a subcomplex of $\check{C}_\delta(Q)$?

$$VR_\varepsilon(Q) \subset \check{C}_{2\varepsilon}(Q)$$

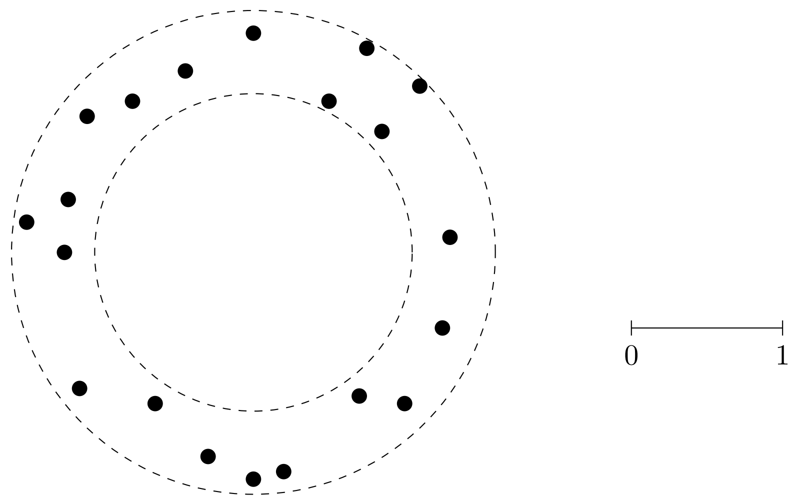
can δ be smaller than 2ε?

(b) For what parameters ϵ and δ is $\check{C}_\delta(Q)$ a subcomplex of $VR_\epsilon(Q)$?

$$\check{C}_\varepsilon(Q) \subset VR_\varepsilon(Q)$$

We did not do this problem:

3. The following point cloud Q was sampled from an annulus.



(a) For approximately what values of ϵ is $VR_\epsilon(Q)$ homotopy equivalent to an annulus?

(b) For approximately what values of ϵ is $\check{C}_\epsilon(Q)$ homotopy equivalent to an annulus?